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A TREATISE ON ELEMENTARY DYNAMICS

A TREATISE ON
ELEMENTARY DYNAMICS

DEALING WITH RELATIVE MOTION
MAINLY IN TWO DIMENSIONS

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BY
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PREFACE.

It is hoped that this little treatise may be of use to Candidates for mathematical scholarships, and to others whose reading is not quite elementary. I have assumed on the part of the student a knowledge of Trigonometry up to the Solution of Triangles, and of the simpler parts of Analytical Geometry; I have made occasional reference to the notation of the Differential Calculus for the benefit of students who may be beginning that subject.

Chapter I. deals chiefly with matter which in one form or another will be familiar to the student; I have, therefore, confined it within the narrowest limits possible. Here, and throughout, I have used the bar, introduced by Maxwell in *Matter and Motion*, to distinguish vector quantities; thus $\bar{u}$ will denote the vector whose scalar magnitude is u; the bar, however, has been employed only when necessary to prevent ambiguity, or to indicate, without circumlocution, that direction must be taken into account; for many purposes the phrase 'a velocity u' is a sufficiently clear abbreviation for 'a velocity whose magnitude is u,' some definite direction being implied by the use of the word *velocity* instead of the word *speed*.

In Chapter II. I have endeavoured to arrange the ideas of Clifford's *Dynamic* (Vol. I.) for the use of ordinary students.

Chapter III. is an attempt to state in an elementary manner the views held by modern writers on the Laws of Motion. It is based mainly on Mach's *Science of Mechanics*

(*Die Mechanik in ihrer Entwickelung*), Clifford's *Dynamic* (Vol. II.), Karl Pearson's *Grammar of Science*, and Mr. W. H. Macaulay's article in the *Bulletin of the American Mathematical Society* (July, 1897); I have throughout verified my references to the *Principia* in Lord Kelvin's Reprint. My object has been to preserve as far as possible the familiar landmarks; this procedure, however, presents difficulties which I can hope at the best to have but partially overcome. I have, in the text, indicated my great indebtedness to Mr. Macaulay; but it seems proper here formally to dissociate him from any responsibility for the contents of this chapter, especially as I cannot be sure that he would regard my statement of the matter as satisfactory. The same remark applies to the section dealing with the Measurement of Time in Chapter I.; but in this section even more than elsewhere I have to assume the responsibility for whatever may be defective.

When my little book was still unfinished, Professor Love's far more able and elaborate work appeared. As I was writing for less advanced students, and, though with the same intention, on quite a different plan, I was the more encouraged to proceed. I have availed myself of Professor Love's work, so far as my ability has permitted me, to check here and there what I have written; I further owe something to it in my treatment of Reactions.

Chapter IV. deals more fully with Work and Energy than has hitherto been customary in elementary books; Dr. Hicks' *Elementary Dynamics*, to which I have once or twice had occasion to refer, is an exception in this respect.

Chapter V. treats the Theory of Dimensions in a simple, and I trust satisfactory, manner.

Chapters VI., VII., and VIII., dealing respectively with Impact, Projectiles, and the Simple Pendulum, are slighter than is usual in text books of this class; but they contain, I hope, all that is essential. Much of the matter sometimes given under these heads finds a place in earlier parts of the present book. The comparatively small bulk of these

chapters is due also to a feeling that more time is devoted, as a rule, to these problems than their intrinsic importance warrants.

For some years past I have been accustomed to show to my pupils the graphic methods which I have occasionally employed in this book; when I began to collect examination papers I speedily became convinced that these methods would be already familiar to many students.

In addition to the works mentioned above, I have from time to time referred to Maxwell's *Matter and Motion*, to Thomson and Tait's *Natural Philosophy*, to Routh's *Rigid Dynamics* (Part I.), to Besant's *Dynamics*, to Garnett's *Elementary Dynamics*, and to Lock's *Elementary Dynamics*.

In selecting examples I have looked through not only Cambridge examination papers, but through many set by examining bodies elsewhere; a few of the examples are my own. I have to thank the various college authorities who have either supplied me with papers or given me access to their libraries. In particular I must tender my acknowledgments to the authorities of University College, Gower Street, for copies of papers set by Professor Pearson, from which I obtained more than one valuable hint; the use of the convenient term *speed-acceleration* I ventured to adopt from these papers.

I trust that I have acknowledged all my manifold obligation to the work of others, or at least that any omission may be put down to oversight. I cannot close these remarks without a few words of gratitude to the friends who have helped me, especially to Mr. Arthur Berry, whose constant kindness alone has encouraged me to proceed; my obligation to him is greater than I can express; at the same time I ought to say that the calls on his leisure are so great that I did not feel it right to ask him to give more than the most cursory glance at the sheets, and I cannot but fear that there may be much in the little book which he might not approve. To my friend Mr. J. T. Little, of Bedford Grammar School, I owe thanks for his continuous care in

reading the sheets and in verifying all the examples; also to my friend and former pupil, Mr. H. A. Webb, Scholar of Trinity College, Cambridge, for much work of the same kind.

My leisure for writing has long been of the scantiest; how scanty, may be inferred from the fact that a rough draft of the whole book with the exception of Chapters IV. and VIII. was already finished in 1896; I can only hope, with no great confidence, that I have avoided serious error. I shall be grateful to correspondents who will draw my attention to any mistakes or inaccuracies which they may discover.

H. A. ROBERTS.

4 MORTIMER ROAD,
CAMBRIDGE, *August*, 1900.

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CHAPTER I.

ON THE MEASUREMENT OF QUANTITY AND THE FUNDAMENTAL UNITS.

1. ASK the question: "How far is it from London to Leeds?" The answer is: "200 miles"; that is, the answer consists of two parts, one a definite length recognised by everybody, and the other a number. In the same way, to measure any quantity whatever, two things are required—(1) a *unit*, that is, a quantity of the same kind as the quantity we are measuring, but of fixed and definite magnitude; (2) a *number* expressing the ratio of the quantity to the unit. This number is called the *measure* of the quantity.

2. There are three kinds of quantity on the measurement of which that of all other physical quantities depends; these are **Length, Time, Mass.**

Two systems of units are employed in England; one, the so-called British (or foot-pound-second) system, the other the C.G.S. (centimetre-gram-second) system; the former is sanctioned by custom, the latter is used in all scientific work; it is also in general use on the continent of Europe.

3. **Unit of Length.** The British unit of length is a foot.

DEFINITION. *A foot is one-third of the distance between the centres of two gold plugs in a certain rod of bronze kept in the Exchequer Chambers, when that rod is at a temperature of* 62° *Fahrenheit.* (62° Fahrenheit is the average temperature of the air in England.)

The C.G.S. unit of length is a centimetre.

DEFINITION. *A centimetre is one-hundredth part of the distance between the ends of a certain rod of platinum kept at Paris, when that rod is at a temperature of* 0° *Centigrade.* (0° Centigrade is the temperature of melting ice.)

4. The idea of measuring **time** seems to arise from the fact that many of our sensations recur in an invariable sequence in easily recognised groups. Such recurring groups of sensations are connected chiefly with the motion of various bodies; we may take, as instances, the apparent daily and annual motions of the Sun, the motion of Jupiter's moons, the vibration of a pendulum in a vacuum, the vibration of an atom of some definite chemical substance, or even the human pulse-beat. Let us call these various groups A, B, C, ...; and while the group A is repeated a times, let the group B be repeated b times, the group C c times, and so on; then when $a, b, c, \ldots$ are all large numbers, it is found that whatever the particular numbers $a, b, c, \ldots$ may be, the ratios $a : b : c \ldots$ are nearly constant, but that some of the ratios are much more nearly constant than others; it is necessary to take a large number of repetitions, because it is exceedingly improbable that a given number of repetitions of one group will correspond *exactly* to a given number of repetitions of any other; but when the numbers are large, one repetition more or less makes but little difference. The best time-measurer would be that recurring group which makes as large a number of the ratios as possible as nearly constant as possible.

A simple instance may make this clear.

Galileo is said to have discovered the constancy of the period of vibration of the pendulum by counting the number of beats of his pulse corresponding to a given number of vibrations of the pendulum; we may imagine him to have proceeded to test one pendulum against others, and one pulse against others, and finally pendulums against pulses; he would have found the ratios of corresponding numbers of vibrations very nearly constant for all the pendulums, much less so for pendulums and pulses, and still less so for pulse and pulse; the pendulum is therefore a better time measurer than the pulse.

The motion used for measuring time is the rotation of the Earth on its axis. Equal intervals of time are taken to be those during which the Earth turns about its axis through equal angles relative to the stars.*

Unit of Time. The unit of time is the same in both systems of units, and is called a second.

DEFINITION. *A second is a definite fraction* ($\frac{1}{86164}$) *of the time taken by the Earth to rotate on its axis relative to the stars.*

When physicists discuss the question of the possible variation in the length of a second, some such comparison of periodic

* A minute correction is necessary. See Herschel's *Astronomy*, § 908, Note.

motions with each other as that contemplated above is held in view.

5. We are not in a position to discuss the nature of **Mass** and the units in which it is measured until we come to consider the Laws of Motion. The student will find the units of **Mass** defined in Chapter III., § 60.

6. On Limits. Suppose that we are measuring some length, such as the distance between two minute scratches on a metal rod, and that after using the most accurate instruments at our disposal, such as measuring microscopes, we assert the length to be 2·7245 feet; what is the precise meaning of this statement? Not that all the decimal places to the right of the 5 are necessarily empty, but that our measuring instruments will not enable us to fill them; even if we obtain better instruments and so fill two more places, the measurement, though nearer the truth, is still not absolutely exact, and no increase in the accuracy of measurement would make it so. The same is true of all measurements; the better our instruments, the more closely can we approximate to the truth, but that is all; even were two lengths by some extraordinary accident *absolutely* the same, we should have no means of ascertaining that this was the case.

Applied Mathematics have always such measurements in view; but in order to make our results applicable to every degree of approximation, however close, the idea of a **limit** is introduced.

Suppose that a certain quantity is being estimated by some process either of calculation or physical measurement, and that the value assigned to it depends on the extent to which the process is carried out.

For example,

(1) The sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\dots$ is $2-\frac{1}{2^9}$ when $n=10$, $2-\frac{1}{2^{99}}$ when $n=100$, $2-\frac{1}{2^{999}}$ when $n=1000$.

(2) The value of $\frac{\sin x}{x}$, where x is the circular measure of an angle, is 0·8... when $x=1$, 0·998... when $x=0·1$, 0·99998 when $x=0·01$, and 0·9999998 when $x=0·001$.

Then if it happens, either from the beginning or after a certain stage in the process, that there is a definite value L such that the difference between L and the quantity continually decreases as the process goes on, and can be made as small as we please by carrying the process far enough, L is said to be the **limit** of the value of the quantity obtained by the process.

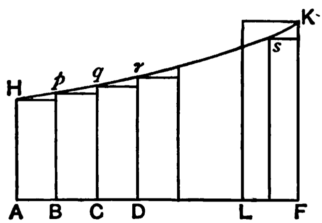
This is sometimes expressed by saying that, as the process proceeds, the quantity *tends to* the limit.

For example, the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots$ tends to the limit 2 when n is indefinitely increased; and $\frac{\sin x}{x}$ tends to the limit unity when x is indefinitely diminished.

In the case of the rod just mentioned we may imagine successive improvements of the measuring process to give 2·7 feet, 2·73 feet, 2·724 feet, 2·7245 feet, and so on. Although we cannot assign any definite number as a limit to the quantity which we call the length of the rod, yet it is evident that as the process proceeds this quantity behaves in a manner analogous to a quantity which is tending to a limit.

7. The following is an example of a limit which will be frequently used in the course of this work:

It is required to find the area of the figure $HKFA$, bounded by the curve HK, the parallel straight lines HA, KF, and the straight line AF.



Divide AF into any number of parts AB, BC, $CD\ldots$, and on them describe the contiguous parallelograms HB, pC, qD, ... sF, each with one corner (viz., H, p, q, etc.) on the curve. Then the area required is equal to the limit to which the sum of the areas of the parallelograms approaches as the divisions AB, BC, $CD\ldots$ are each diminished indefinitely, and the number of divisions consequently increased indefinitely.

For let LF be equal to the greatest of the divisions AB, BC, ... The difference between the required area and that of the sum of the parallelograms can be shown to be less than the area of the parallelogram KL. But the area of this parallelogram can be continually diminished and made less than any assignable finite area by sufficiently diminishing the divisions AB, BC, $CD\ldots$, etc. Hence the required area is the limit of the sum of the areas of the parallelograms.

Elementary Theorems on Vectors.

8. **Position.** The position of a point in space can only be determined relative to other points.

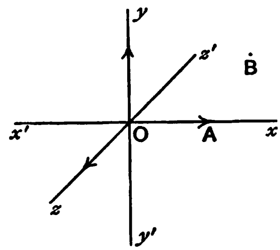
If P and Q are two points, the position of Q relative to P

is known when the *magnitude* of the straight line PQ is known, and also its *direction* relative to the system of points considered.

The magnitude and direction of PQ may be most simply indicated thus:

Let a point travelling along the straight line PQ from P to Q be said to describe the straight line in the *positive sense*, and let the positive senses of all other straight lines be similarly defined.

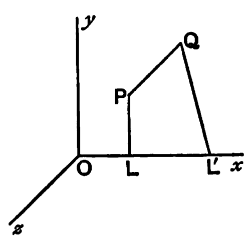
Take any three points of the system O, A, B. Join OA, and take this line for *axis of x* as in coordinate geometry, the *positive sense* being O to A.



Draw yOy' perpendicular to Ox *in* the plane OAB, and zOz' perpendicular to this plane, and let the positive senses of these be Oy, Oz, and be so chosen that if any two of the axes Ox, Oy, Oz be supposed turned round the third in such a sense that Ox follows Oy, Oy follows Oz, or Oz follows Ox, the sense of turning is related to the positive sense of the third axis in the same way as the senses of twisting and travelling in an ordinary or right-handed screw.

The reader will find that, given the positive senses of Ox and Oy, that of Oz is determinate.

Let LL', MM', NN' be the *orthogonal projections* (see Hobson's *Trigonometry*, § 17) of PQ on the axes. As a point describes PQ, the point which is its projection on Ox describes LL'; if LL' is described in the positive sense of Ox, let the projection LL' be called positive, otherwise negative, and let the senses of the projections MM', NN' be similarly determined.



Then when the magnitude and sense of the projections LL', MM', NN' are given, the magnitude and direction of PQ is given.

The word *direction* must be taken to include *sense*.

Note that to determine axes of coordinates *three* points of reference are required in space of *three* dimensions, or *two* in space of *two* dimensions. Also when the axes are determined

the position of Q relative to P is determined by *three* numbers in the former or *two* in the latter case.

9. Vectors and Scalars. Any quantity which is completely determined when its magnitude and direction are known is called a *directed* quantity or **vector.**

Any quantity involving magnitude only is called a **scalar.**

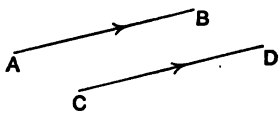
In particular the straight line PQ, given in magnitude and direction with positive sense P to Q, is called the **position vector** of Q relative to P.

Any directed quantity may be represented by a straight line, the number of units in whose length (on some convenient scale) is equal to the number of units in the quantity, and whose direction is the direction of the quantity. The sense of the vector may be indicated by an arrowhead on the straight line which represents it.

As we see from the preceding paragraph two scalars are sufficient to determine a vector in space of two dimensions, or three in space of three dimensions.

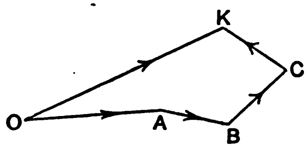
The exact point from which the line representing a vector starts is to be regarded as immaterial unless the contrary is stated, provided only its length and direction are given.

Thus if A, B, C, D are four points on a map such that B is two feet north-east of A and D two feet north-east of C, the position vector of B relative to A is regarded as being the same as that of D relative to C; what is meant by this identity is that we go through the same process to arrive at B from A as to arrive at D from C. In each case we travel two feet to the north-east.



10. Vector Sum.

Let there be any number of vectors α, β, γ, ... of the same kind; starting from any point O, draw a straight line OA to represent α; from A draw AB to represent β, from B draw BC to represent γ, and so on; let K be the terminal point of the incomplete polygon so formed; then the vector represented by OK is called the *Vector Sum* of α, β, γ, ...



Resultant of any number of Vectors. If α, β, γ, ... be any number of vectors of the same kind, and if a single vector ρ

can be found whose effect (whatever that may be) is exactly equivalent to that of α, β, γ, ... combined, then the vector ρ is called the *Resultant* of the vectors α, β, γ, ...

The combination of vectors of every kind considered in this treatise is governed by one rule, which will be proved for vectors of various kinds as they present themselves, and which may now be stated thus:

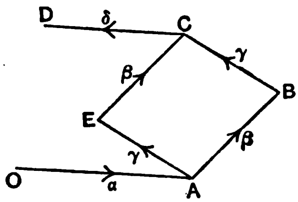
*The resultant of any number of vectors of the same kind is their vector sum.**

For example, if OA, AB, BC are position vectors, the position vector of C relative to O is the vector sum of that of C relative to B, that of B relative to A, and that of A relative to O

11. The following proposition is important:

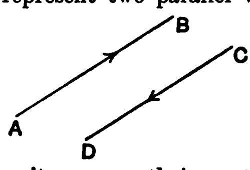
Vector addition is commutative; that is, the vector sum is independent of the order in which the vectors are taken.

For if OA, AB, BC, CD represent the vectors α, β, γ, δ, complete the parallelogram $ABCE$.



AE is equal and parallel to BC, and therefore represents the vector γ; similarly EC represents the vector β. Thus the vector sum of α, γ, β, δ is the same as that of α, β, γ, δ. By interchanging successive pairs we can arrive at any assigned order for adding the vectors, the sum remaining unchanged.

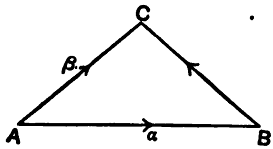
12. If AB, CD represent two parallel vectors of the same



magnitude but opposite senses, their vector sum is *zero*. If then AB be denoted by α, we may denote CD by $-\alpha$.

* There *are* quantities sometimes called vectors to which this rule does not apply;—*e.g.* finite rotations of a rigid body.

Now let AB, AC represent two vectors α, β. BC is the vector sum of BA, AC, that is, of $-\alpha$ and β. BC may thus



consistently be called the *vector difference* of α from β; similarly CB is the vector difference of β from α.

13. If all the vectors are parallel, the magnitude of the vector sum is the *algebraic sum* of the magnitudes of the vectors.

Consistently with this, if m be a pure number, positive or negative, integral or fractional, $m\alpha$ is to be regarded as a vector parallel to α, but of m times its length, in the same sense as α if m be positive, and in the opposite sense if m be negative.

Multiplication by a numerical factor, positive or negative, integral or fractional, may be distributed over vector additions and subtractions; that is,

$$m(\alpha+\beta+\gamma+\ldots)=m\alpha+m\beta+m\gamma+\ldots.$$

To prove this the following proposition suffices:

The vector sum of $m\alpha$, $m\beta$ is m times the vector sum of α and β.

This the student may prove by similar triangles.

Let the vectors α, β, $m\alpha$, $m\beta$ be represented by AB, BC, $A'B'$, $B'C'$, respectively. Prove that the triangles ABC, $A'B'C'$ are similar, and therefore that $A'C'$ is parallel to AC and numerically equal to mAC.

The theorem can be at once extended to the sum of any number of vectors.

14. Having shown that the commutative and distributive laws hold for the addition and subtraction of vectors and their multiplication by scalars, we may use the ordinary signs + and − to denote vector addition and subtraction, *provided we interpret the equality sign (=) to mean "are equivalent as vectors to."*

When *vector addition* is meant, the symbols added will be distinguished as vectors by a *bar* drawn above them.

Thus when it is required to distinguish the vector OA from the scalar OA we shall write the former $\overline{OA}$.

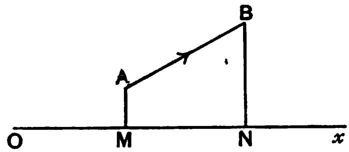
We may then write the symbolic statement of a vector sum thus:

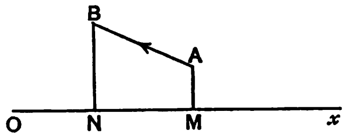
$$\overline{OA}+\overline{AB}+\overline{BC}=\overline{OC}.$$

Greek letters may be used to denote vectors where no ambiguity is possible. The statement $\alpha=\beta$ will then mean that the vector α is equal and parallel to the vector β.

15. DEFINITION. *The angle between two vectors is the angle between their positive senses.*

Resolved part of a vector in a given direction.



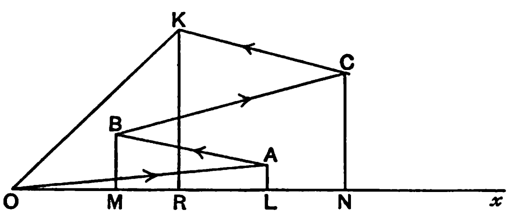


Let Ox (positive sense O to x) be parallel to the given direction, and let AB represent a vector not necessarily in one plane with Ox. Then the *orthogonal projection of AB on Ox* represents a vector called the **resolved part** of the vector AB parallel to Ox.

Let the length of AB be denoted by a, and let the angle between AB and Ox be θ. Then the length of the resolved part of AB parallel to Ox is $a \cos\theta$. When $\cos\theta$ is negative, $a\cos\theta$ is negative, and the sense of the resolved part is x to O.

If we take rectangular coordinate axes, as in § 8, it is evident that *any vector is equal to the vector sum of its resolved parts parallel to the axes.*

PROPOSITION. *The algebraic sum of the resolved parts of a number of vectors parallel to any line is equal to the resolved part of their vector sum parallel to that line.*



Let OA, AB, BC, CK represent the vectors, Ox the given line Draw AL, BM, CN, KR perpendicular to Ox. OL, LM, MN, NR

are the resolved parts. Their algebraic sum is OR; but this is the resolved part of OK the vector sum. The proposition is therefore established.

16. Mean Vector and Mean Point.

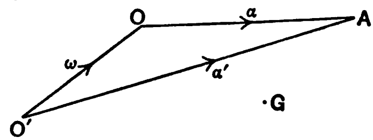
DEFINITION. *If α, β, γ ... be any vectors of the same kind, and* m_1, m_2, m_3 ... *any scalar numbers, integral or fractional, the vector* $(m_1\alpha+m_2\beta+m_3\gamma+\ \dots)/(m_1+m_2+m_3+\ \dots)$, *or* $\frac{\Sigma m\alpha}{\Sigma m}$, *is called the* **mean vector** *of α, β, γ ... for the multiples* m_1, m_2, m_3 ....

The mean vector is, of course, the same in whatever order the addition in the numerator is made.

If m_1, m_2, m_3 ... are all positive integers, the mean vector of the m_1 vectors α, the m_2 vectors β, the m_3 vectors γ ... is obtained by dividing the vector sum of these by the number of the vectors.

DEFINITION. *Let* OA, OB, OC ... *be the position vectors of points* A, B, C ... *relative to a point* O *which may be called the origin; and let* G *be a point such that its position vector* OG *relative to* O *is the mean vector of* OA, OB, OC ... *for the multiples* m_1, m_2, m_3 .... *Then* G *is called the* **mean point** *or* **centroid** *of* A, B, C ... *for the multiples* m_1, m_2, m_3 ....

PROPOSITION. *The position of the mean point is not affected by altering the origin.*



For, take another origin O'. Denote the vector $O'O$ by ω, the vectors OA, OB, OC ... by α, β, γ, ... and the vectors $O'A$, $O'B$, $O'C$ by α', β', γ' ....

Then G being the mean point for origin O,

$$\overline{OG}=\frac{m_1\alpha+m_2\beta+m_3\gamma+\ \dots}{m_1+m_2+m_3+\ \dots};$$

$$\therefore\quad \overline{O'G}=\omega+\overline{OG}=\omega+\frac{m_1\alpha+m_2\beta+m_3\gamma+\ \dots}{m_1+m_2+m_3+\ \dots}$$

$$=\frac{m_1(\omega+\alpha)+m_2(\omega+\beta)+m_3(\omega+\gamma)+\ \dots}{m_1+m_2+m_3+\ \dots}.$$

But $\omega+\alpha=\alpha'$, $\omega+\beta=\beta'$, etc. ... ;

$$\therefore \quad \overline{O'G}=\frac{m_1\alpha'+m_2\beta'+m_3\gamma'+\ldots}{m_1+m_2+m_3+\ldots},$$

or G is also the mean point for origin O'.

This result may clearly be stated as follows: *whatever be the position of the point* O, *the sum of the vectors* $\mathrm{m_1 . OA}$, $\mathrm{m_2 . OB}$, $\mathrm{m_3 . OC}$ ... etc. ... *is* $(\mathrm{m_1+m_2+m_3+}\ldots)\,\mathrm{OG}$, *where* G *is the mean point of* A, B, C ... *for the multiples* $\mathrm{m_1}$, $\mathrm{m_2}$, $\mathrm{m_3}$ ... .

If (x_1, y_1, z_1), (x_2, y_2, z_2) ... etc. ... be the Cartesian co-ordinates of the various points A, B, C ... referred to rectangular axes Ox, Oy, Oz; then since the resolved part of a vector sum is equal to the algebraic sum of the resolved parts of the vectors constituting it, the coordinates of the mean point of A, B, C ... for the multiples m_1, m_2, m_3 ... are

$$\frac{m_1x_1+m_2x_2+m_3x_3+\ldots}{m_1+m_2+m_3+\ldots}, \text{ or } \frac{\Sigma mx}{\Sigma m},$$

with similar expressions in y and z.

The following important properties of the mean point we leave to the student as exercises:

(1) If there be only *two* points A and B, their mean point for the multiples m_1, m_2 lies on AB and divides it in the ratio $m_2 : m_1$; if $m_1=m_2$, it is the middle point of AB.

(2) The mean point of A, B, C ... for the multiples m_1, m_2, m_3 ... can be obtained thus: take G_1 the mean point of A, B for the multiples m_1, m_2; then take G_2 the mean point of G_1, C, for the multiples m_1+m_2 and m_3, and so on. The last point so obtained is the mean point of the system.

(3) If G be the mean point of the system, the vector sum of m_1GA, m_2GB, m_3GC ... is zero.

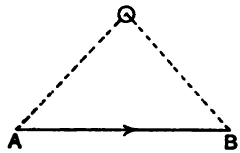
17. It is sometimes convenient to regard the straight line representing a vector as drawn from a particular point. The vector is then said to be *localised* at that point, or to be *initial* at that point.

An important case in which the vectors are localised is the **Theorem of Moments.** This we shall now discuss for vectors in a given plane.

DEFINITION. *The moment of a localised vector about any point is the product of the magnitude of the vector and the length of the perpendicular from the point on the line in which the vector lies.*

Let O be a point in the plane of the vectors, AB a line representing one of them. Then the moment of the vector AB

about O is represented by twice the area OAB; it is reckoned positive when a person, placed on an arbitrarily chosen side of

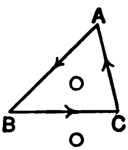


the plane, looking along the vector in its positive sense sees the area on his left, negative if he sees it on his right.*

PROPOSITION. *The algebraic sum of the moments of three coinitial vectors whose sum is zero about any point in their plane is zero.*

Let AB, BC, CA represent three vectors whose sum is zero, and let O be any point in their plane.

First suppose that the vectors are initial at A, B, C respectively.



Then the algebraic sum of their moments about O is equal to $\pm$ (twice the area of the triangle ABC).

The vector AB is initial at A.

The moment of the vector CA would be unaltered by supposing it to become initial at A.

The algebraic sum of the moments would be diminished if positive, and increased if negative, by twice the area ABC if the vector BC were replaced by a vector of the same magnitude, direction, and sense, but initial at A.

Hence if the three vectors AB, BC, CA represent in magnitude, direction and sense, three vectors coinitial at A, the algebraic sum of the moments of these three is zero.

Cor. I. *The algebraical sum of the moments of any number of coinitial coplanar vectors whose sum is zero about any point in their plane is zero.*

The proof of this we leave as an exercise to the student.

Cor. II. Defining the moment of a localised vector about an axis as the moment of its orthogonal projection on a plane

* Areas, and therefore moments, are in reality vector quantities, which may be represented by straight lines of given length and sense perpendicular to their planes. The vector sum of a number of coplanar areas is numerically equal to their algebraic sum.

perpendicular to the axis about the point in which the axis cuts this plane, it is evident that

The algebraical sum of the moments of any number of coinitial vectors, not necessarily in one plane, whose sum is zero, about any axis, is zero.

Examples on Chapter I.

1. $ABB'A'$ is a quadrilateral. Prove that the vector difference of $A'B'$ from AB is equal to the vector difference of BB' from AA'.

2. Prove that the straight lines which join the middle points of opposite edges of a tetrahedron meet in a point.

3. If G be the point of concourse of the medians of a triangle, O a point either in or outside its plane, prove that OG is the mean vector of OA, OB, OC for equal multiples. Prove also that the vector sum of GA, GB, GC is zero.

4. P is a point such that the *length* of the resultant of the position vectors PA, PB, PC is constant, A, B, C being fixed points. Find the locus of P.

5. If there are any two groups of points A, B, C ... and A', B', C', ..., prove that the mean points of the first system (multiples m_1, m_2, m_3, ...), of the second system (multiples m_1', m_2', m_3', ...), and of the combined system (multiples m_1, m_2, m_3 ... m_1', m_2', m_3', ...) are collinear.

In what ratio does the mean point of the combined system divide the line joining the mean points of the separate systems?

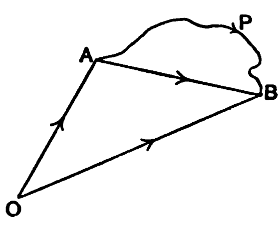
CHAPTER II.

KINEMATICAL THEOREMS.

18. KINEMATICS is the name given to that portion of mathematical science which teaches us to describe motion. It forms the introduction to Dynamics proper, which treats of motion of bodies as affected by other bodies. Dynamics again is conveniently divided into Kinetics and Statics, the former treating of the circumstances of relative motion, the latter of the circumstances of relative rest. It is usual, and on the whole convenient, to reserve the subject of Statics for treatises dealing with it alone.

19. Displacement.

DEFINITION. *The displacement of a point* P *relative to a point* O *is the change of* P*'s position vector relative to* O.



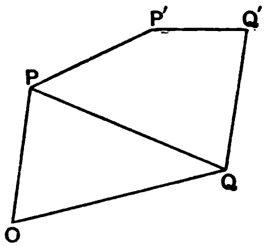
Thus if a point P move from A to B *by any path whatever*, AB the vector difference of OA from OB is the displacement of P relative to O.

PROPOSITION. *If a point* P *undergoes a displacement* α *relative to* O, *while* Q *undergoes a displacement* β *relative to* P, *the displacement of* Q *relative to* O *is the vector sum of* α *and* β.

Let OP, OQ be the position vectors of P, Q relative to O *before*, OP', OQ' *after* the displacement.

Then Q's displacement relative to O

$$=\overline{QQ'}=\overline{QP}+\overline{PP'}+\overline{P'Q'}=\overline{PP'}+\overline{P'Q'}-\overline{PQ}.$$



But $\overline{PP'}$ is P's displacement relative to O, that is, α; and $\overline{P'Q'}-\overline{PQ}$ is the change of Q's position vector relative to P, that is, Q's displacement relative to P, or β. Thus Q's displacement relative to O is the vector sum of α and β.

Corollary. If the position vector of Q relative to O is the sum of the vectors α_1, α_2, α_3, ... and these are changed to $\alpha_1+\delta_1$, $\alpha_2+\delta_2$, $\alpha_3+\delta_3$, ..., Q's displacement relative to O is the vector sum $\delta_1+\delta_2+\delta_3+\ldots$.

Returning to the proposition, note that if P be displaced β relative to O and Q be displaced α relative to P, the displacement of Q relative to O is the same as before, viz., the vector sum of α and β.

Finally, the displacement of Q is in each case the same as if it had undergone two *separate* displacements α, β, *each relative to* O. The importance of these remarks will appear later.

In all cases the displacement of a point Q relative to a point P is equal and opposite to that of P relative to Q, for the vector sum of the two displacements is, by the above proposition, zero.

On Velocity.

20. Definition. *The velocity of a point* P *relative to a point* O *is the time-rate of* P*'s displacement relative to* O.

Thus, just as displacement is the change of a vector OP, velocity is the rate of change of a vector OP.

Velocity, as is implied in the definition, is *relative only*. In speaking of the velocity of a point, we can only mean the rate at which its position estimated with regard to other points is changing, that is, we mean its velocity *relative* to those points.

So, to say that a point is "*at rest*" only implies that its position relative to other points is not changing; this is the only scientific meaning which can be attributed to the phrase "*at rest.*"

When the points or axes to which the motion is referred are clearly understood, the word "*velocity,*" used without qualification, is used to denote velocity relative to those points or axes. Thus in dynamical problems the velocity of a point relative to certain axes through a given point on the Earth's surface is often for brevity called the *velocity* of the point.

21. Constant Velocity.

DEFINITION. *The velocity of a point is constant when equal displacements take place in* **any two** *equal times.*

The phrase "equal displacements" implies that the displacements must take place in the same direction and be equal in magnitude. Thus a point moving with constant velocity must travel along a straight line.

Constant velocity is a vector quantity.

Its magnitude = the magnitude (number of feet or centimetres) of the displacement which takes place in a second. Its direction is the direction of the displacement. The magnitude of a velocity is called its *speed.*

A point may travel with constant speed along any path whatever; it will describe equal lengths of the path in any two equal times.

If two points describe the same path in opposite senses, their speeds will be distinguished by opposite algebraic signs.

The *Unit Speed* is the speed of a point which passes with constant speed over a foot in a second (or in the C.G.S. system over a centimetre in a second).

These units are for brevity denoted by 1 f.s., and 1 cm.s. respectively.

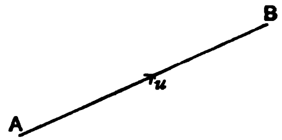
A point moving with unit speed in a given direction is said to have unit velocity in that direction.

A velocity whose magnitude is u is often spoken of as "a velocity u," it being understood from the use of the word *velocity* that some definite direction is implied; when a vector-symbol is required to denote the velocity whose magnitude is u, we shall employ $\bar{u}$.

22. PROPOSITION. *If a point travels with constant velocity* u *feet a second for* t *seconds, its displacement is equal to* ut *feet.*

Let A be the initial position of the point, and let AB be the direction of the velocity u.

Because the velocity is constant, the point will travel in the straight line AB.



Then, if t be a whole number, since by definition (§ 21) the point's displacement is u feet in each second, it is $u \times t$ or ut feet in t seconds.

If t be a fraction, let it be equal to $\frac{p}{q}$, where p and q are integers. Since in each $\frac{1}{q}$ of a second the point must by definition have the same displacement, the displacement is $\frac{u}{q}$ in $\frac{1}{q}$ of a second, and therefore $p \cdot \frac{u}{q}$, or $t \cdot u$ feet in $\frac{p}{q}$ seconds, as before.

If O be the point relative to which the velocity of the moving point is estimated, and if OA, OB be the position vectors of the moving point at the beginning and end of the interval t, then $\overline{OB} = \overline{OA} + \bar{u}t$.

Variable Velocity.

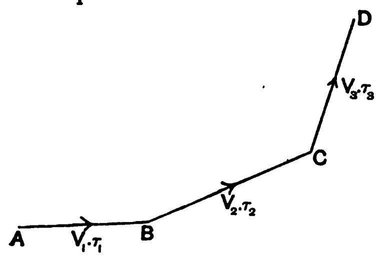
23. Constant velocity may be estimated in an interval as small as we please; *e.g.* a point displaced with constant velocity through 1 foot in a second will be displaced through $\frac{1}{10^6}$ feet in $\frac{1}{10^6}$ seconds in the same direction, and however small the interval of time is taken, (1) the direction of displacement, (2) the speed will be the same; this is a direct result of the definition that equal displacements take place in *any two* equal times. Thus *the velocity at any instant may be determined by estimating the displacement during any interval containing that instant.*

But suppose that the moving point is travelling in any way whatever other than along a straight line with constant speed. If we seek to define the velocity at a given instant, neither of the above statements (1), (2) is in general true. For if we consider the displacement for an interval τ, which includes the given instant, and diminish τ, the ratio $\left(\frac{\text{displacement}}{\tau}\right)$ will alter in general both in magnitude and direction.

However, in the majority of cases, the ratio $\left(\frac{\text{displacement}}{\tau}\right)$ tends to become equal to a definite vector, as τ diminishes indefinitely.

DEFINITION. *When a point is moving in any manner whatever, the ratio* $\left(\frac{\textit{displacement in a given interval } \tau}{\tau}\right)$ *is called the* **average velocity** *for that interval.*

It is that constant velocity which will produce the *actual* displacement of the point in time τ.



If we are given V_1, V_2, V_3, ..., the average velocities of a point for a number of successive intervals τ_1, τ_2, τ_3, ..., and if we draw the successive displacements $AB = V_1\tau_1$, $BC = V_2\tau_2$, $CD = V_3\tau_3$, ... we obtain not, it is true, the path of the moving point, but a number of points B, C, D, ... on the path, and $ABCD$... is a polygon inscribed in the path.

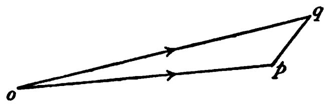
24. We shall now define the **velocity of a point at any instant** of its motion.

DEFINITION. *If the average velocity of a point for an interval* τ *including the particular instant can be made to approach as nearly as we please to a definite vector by diminishing* τ *indefinitely, the point is said to have a velocity* **at the instant**, *and its value is the limit to which the average velocity* $\left(\frac{\textit{displacement}}{\tau}\right)$ *tends.*

The interval τ may be chosen so that the instant coincides with either end of it or falls between them ; it may happen that the results obtained by thus varying the position of the interval with regard to the instant are different. In this exceptional case the point cannot be said to have a velocity at the instant.

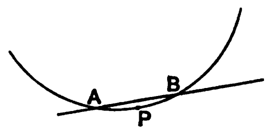
Suppose that oq represents the average velocity for the interval τ, and op the definite vector to which it approaches ;

the definition implies that the vector pq can be made shorter than any assignable length, however small, by sufficiently diminishing τ.



It is sometimes convenient to call the velocity of a point at a given instant the *instantaneous velocity* of the point.

Let A and B be two points of the path, one on each side of a point P. If the moving point has an instantaneous velocity at P, as we suppose A and B to move up to one another, the chord AB, being the direction of the average velocity from A to B, tends by our definition to a definite direction, viz. that of the *tangent* at P according to Newton's definition of a tangent. Accordingly when the moving point has an instantaneous velocity at a point P of its path, there is a tangent to the path at P, and this tangent is the direction of the instantaneous velocity.



The reader acquainted with the elements of the differential calculus will see that the *speed* of a point at a given instant may be represented by $\frac{ds}{dt}$, where s is the length of the arc of the path measured from a given point. Newton denoted the time-rate of the increase of a quantity by a dot placed over the symbol representing the quantity; thus in his notation the speed of a point may be written $\dot{s}$, while if ρ be the point's position vector relative to a point O, its velocity relative to O may be written $\dot{\rho}$, a vector symbol.

For a fuller discussion of instantaneous velocity see Clifford's *Dynamic*, Vol. I., pp. 44-52, which work the reader should at some time consult.

25. The velocity of a point at any instant of the motion is, as we have just seen, represented by a *definite vector*, that is, it is equivalent to a *certain constant velocity*. We will give two illustrations of this important aspect of a variable velocity, taking for simplicity cases in which the speed alone varies.

Suppose that two trains A and B are travelling in the same sense on parallel lines of rail, and suppose that A starts from rest and gradually increases its speed up to 20 miles an hour,

while B moves with a constant speed of 10 miles an hour. At first A will appear to lose on B, but when A has attained its greatest speed it will be gaining on B; *there must then be some instant at which A stops losing on B and begins to gain. At that instant the train A has a speed of* 10 *miles an hour.*

Again, in the case of a stone falling vertically the distance traversed in t seconds from rest is given (as will be proved in § 40) by $s=at^2$, where a is the distance traversed in the first second. Suppose another body to move vertically downwards with constant speed v, starting with the stone. The distance between the bodies at time t

$$=vt-at^2=\frac{1}{4a}\{v^2-(v-2at)^2\}, \text{ by algebra.}$$

This quantity continually increases as t increases, until $v-2at=0$, or $t=\frac{v}{2a}$, after which it diminishes. Hence at the instant when $t=\frac{v}{2a}$ each body is *at rest* relative to the other. Consequently the speed of the falling body at the end of a time t (which may be made of any duration by properly choosing v) is $2at$.

The above illustrations are taken substantially from Clifford's *Dynamic.*

In a similar way, the instantaneous velocity of a point moving in any manner whatever may be regarded as equal to a certain constant velocity; we have only to make the speeds agree as above, and then to make the directions agree.

Examples.

1. When the velocity is constant in direction, and the displacement of the moving point at time t is equal to at^2, find, from the definition in § 24, the speed at time t.

[Displacement in time $t=at^2$, displacement in time $t+\tau=a(t+\tau)^2$;

$\therefore$ Displacement in interval $\tau=2at\tau+a\tau^2$;

$\therefore$ The ratio $\frac{\text{displacement}}{\tau}=2at+a\tau$, which approaches $2at$ as τ is indefinitely diminished.]

2. Under the same circumstances, find the speed at time t when the displacement at time t is at^3.

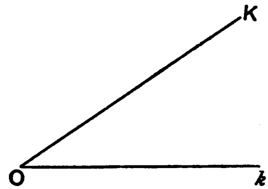
3. The velocity still being constant in direction, find the speed at time t, when the displacement at time $t=a\sin pt$.

4. A point describes a circle of radius 1 foot with a constant speed of 1 foot per second. Another point describes a regular polygon of 100 sides inscribed in the circle, both revolutions being accomplished in the same time. Prove that the difference in their speeds is ·00016 f.s. nearly. Also if they reach an angle of the polygon together, and the first point then travels with unaltered speed along the tangent to the circle, prove that when the second has described the next side of the polygon, the points are less than $\frac{1}{30}$ of a side of the polygon apart.

5. Prove by the method of § 25 that if the displacement of a moving point at any time t be $s = \frac{1}{3}a^2t^3$, the direction being constant, the speed at time $t = a^2t^2$.

[The distance between the points moving with variable and constant speed $= \frac{1}{3a}\{2v^{\frac{3}{2}} - (2v^{\frac{3}{2}} - 3avt + a^3t^3)\}$. Then prove that the last term is positive, except when $v = a^2t^2$.]

26. Proposition. *If a point has a velocity at each instant of its motion, its path may be represented to any required degree of accuracy by supposing the point to travel for a short interval τ with the velocity it has at the beginning of this interval, for another short interval τ with the velocity it has at the beginning of this second interval, and so on.*



Let $\bar{v}_1, \bar{v}_2, \bar{v}_3 \ldots$ be the velocities at the beginnings of the intervals, $\overline{V}_1, \overline{V}_2, \overline{V}_3 \ldots$ the *average* velocities for each interval, O the initial, K the final position of the moving point, k the final position on the above hypothesis. Let t be the time from O to K, and let the intervals τ be n in number. Then $n\tau = t$.

Then (definition of average velocity, § 23)

$$\overline{OK} = \overline{V}_1 . \tau + \overline{V}_2 . \tau + \overline{V}_3 . \tau + \ldots$$

$$= (\overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \ldots)\tau ;$$

But $$\overline{Ok} = (\bar{v}_1 + \bar{v}_2 + \bar{v}_3 + \ldots)\tau ;$$

$$\therefore \quad \overline{kK} = [(\overline{V}_1 - \bar{v}_1) + (\overline{V}_2 - \bar{v}_2) + \ldots]\tau.$$

Now let $\bar{u}$ be the greatest of the vectors $\overline{V}_1 - \bar{v}_1$, $\overline{V}_2 - \bar{v}_2$, etc.

$\therefore$ kK is numerically less than $nu\tau$ or ut.

But u, and therefore ut, may be made to diminish indefinitely by sufficiently diminishing τ (Definition, § 24). Whence the proposition.

The motion of the particle need not be confined to one plane.

Corollary. The same would be true if the point were supposed to travel throughout each interval with the velocity at the *end* of the interval, or with the velocity at any other point of the interval.

The intervals during which the point is supposed to travel with constant velocity need not of necessity be all equal; it is sufficient if they all diminish indefinitely. The proof of this we leave to the student.

Examples.

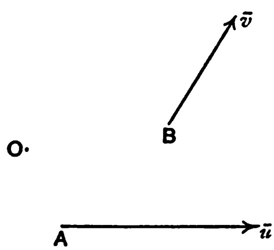
1. A point describes a circle with any constant speed in 4000 seconds. Prove that if it be supposed to travel for each second with the velocity it has at the beginning of that second, at the end of 1000 seconds its displacement from the position it would have occupied at that time if it had continued to travel in the circle is slightly more than $\frac{1}{1000}$ of the radius.

2. A point moves with a velocity constant in direction; the velocity is initially 8 cm.s., and at any subsequent time t is $(8+t)$ cm.s. Prove, by considering the velocities at the beginning and end of intervals each $\frac{1}{10}$ of a second, that in 10 seconds the space described is greater than 129·5 and less than 130·5 cm.

Resultant of Several Velocities.

27. The following is a form of the proposition known as the **Parallelogram of Velocities.** Taken in conjunction with § 29 below it gives explicitly all that is implied in shorter statements of that proposition.

PROPOSITION. *Let* O, A, B *be any three points, and let* A *have any motion whatever relative to* O, *and let* B *have any motion whatever relative to* A; *at each instant of the motion, the velocity of* B *relative to* O *is the vector sum of the velocities of* B *relative to* A *and of* A *relative to* O.



(1) Let the velocities be constant.
Let A's velocity relative to O be $\bar{u}$.
Let B's „ „ A be $\bar{v}$.
Then in a second
B's displacement relative to A is $\bar{v}$, and
A's „ „ O is $\bar{u}$.
∴ B's displacement relative to O is $\bar{u}+\bar{v}$.

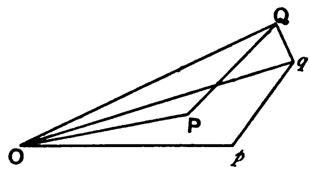
Next, in any interval t,
B's displacement relative to A is $\bar{v}t$;
A's „ „ O is $\bar{u}t$;

∴ B's displacement relative to O is $\bar{u}t+\bar{v}t=(\bar{u}+\bar{v})t$, (§ 13).
that is, $t\times$(displacement of B relative to O in a second).

Therefore B is displaced relative to O with a constant velocity equal to the vector sum of $\bar{u}$ and $\bar{v}$.

(2) Next, let the velocities be variable.

Let OP, PQ represent the *average velocities* of A relative to O and B relative to A for a small interval τ including a given instant. Then OQ represents the *average velocity* of B relative to O; for OP, PQ represent constant velocities for a given value of τ, and therefore the average velocity of B relative to O is their vector sum by the first part of the proof.



Let Op, pq represent the instantaneous velocities of A relative to O and of B relative to A. Then by hypothesis, the vector difference OP from Op (that is Pp), and also the vector difference PQ from pq each diminish indefinitely as τ is diminished.

Call the former α, the latter β.

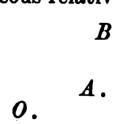
Then
$$\begin{aligned}\overline{Qq}&=\overline{QP}+\overline{Pp}+\overline{pq}\\&=\overline{Pp}+\overline{pq}-\overline{PQ}\\&=\alpha+\beta.\end{aligned}$$

This is a vector whose length is numerically less than the arithmetic sum of the lengths of α and β, and which therefore diminishes indefinitely with τ.

Therefore OQ can be made to approach as nearly as we please to the definite vector Oq by sufficiently diminishing τ.

Therefore Oq is the instantaneous velocity of B relative to O.

But Oq is the vector sum of Op, pq the instantaneous relative velocities, which proves the proposition.

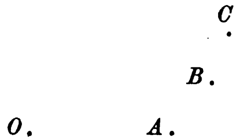


Corollary. Let the points O and B coincide. Then the vector sum of B's velocity relative to A and A's velocity relative to B is B's velocity relative to B, that is, zero.

Therefore the velocity of A *relative to* B *is equal and opposite to that of* B *relative to* A.

Referring back to § 24, we can now interpret the meaning of pq, the vector difference between the *average* and *instantaneous* velocities thus—let one point be moving with the *average* velocity for a given interval, and another point with the *instantaneous* velocity at some instant of it; then pq represents the velocity of one of these points relative to the other.

28. Polygon of Velocities. The proposition of § 27 may be easily extended to the motion of any number of points. Denote these points by $O, A, B, C, \dots$.



Let $\bar{u}_1$ denote the velocity of A relative to O,

$\bar{u}_2$ that of B relative to A,

$\bar{u}_3$ that of C relative to B, and so on.

Then the velocity of the last point K relative to O is the vector sum of $\bar{u}_1, \bar{u}_2, \bar{u}_3, \dots$.

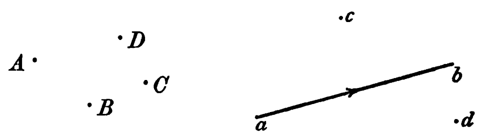
For the velocity of B relative to O is the vector sum of $\bar{u}_1, \bar{u}_2$. The velocity of C relative to O is the vector sum of the velocity of B relative to O, and that of C relative to B, that is, of $\bar{u}_1, \bar{u}_2, \bar{u}_3$, and so on for any number of points.

Let $\alpha, \beta, \gamma, \dots$ denote the position vectors of A relative to O, B relative to A, etc., so that the position vector of the last point K relative to O is $\alpha+\beta+\gamma+\dots$; the velocity of any point C relative to a point B is the rate of change of the corresponding vector γ, or in Newton's notation $\dot{\gamma}$. We may thus write the velocity of K relative to O as the vector sum $\dot{\alpha}+\dot{\beta}+\dot{\gamma}+\dots$.

The above result can be graphically represented as follows:

Let $A, B, C, \dots$ be any number of points, moving with various velocities relative to one another.

If we draw from any point the vector ab to represent B's velocity relative to A, bc to represent C's velocity relative to B, and so on, we have an assemblage of points $a, b, c, d, \dots$ which is called the **velocity diagram.**



To represent the relative velocity of two given points we have only to join the corresponding points in the velocity diagram.

Example. A steamer is travelling with constant velocity in a steady breeze; find the direction of the line of smoke from the funnel.

Let oa represent the velocity of the steamer, ob that of the wind. Assuming that the smoke on leaving the funnel at once takes up the velocity of the wind, ab represents the velocity of a particle of the smoke relative to the funnel. Hence the direction of the smoke-line is ab.

29. Independent Velocities. The vector sum of the quantities $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, ... is independent of the order of addition. Hence, given any series of points O, A, B, C, D, E, the velocity of E relative to O will be the same in whatever order the relative velocities $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, ..., are attributed to the pairs of points; for example, we shall obtain the same vector for the velocity of E relative to O, whether we call $\bar{u}_5$ the velocity of E relative to D, $\bar{u}_4$ the velocity of D relative to C, and so on, or whether we call $\bar{u}_2$ the velocity of E relative to D, $\bar{u}_5$ the velocity of D relative to C, and so on; of course in the two cases the velocities of the intermediate points D, C, B, A relative to O will not be the same.

In many cases the particular network of connecting points used is quite unimportant. We proceed to discuss a nomenc'ature which will do away with the necessity for specifying them.

Consider, for simplicity, the case of two constant relative velocities, $\bar{u}_1$ that of A relative to O, and $\bar{u}_2$ that of B relative to A, and let the corresponding displacements of A relative to O, and B relative to A, each in an interval t, be α and β, so that $\alpha = \bar{u}_1 . t$, $\beta = \bar{u}_2 . t$.

Now the displacement of B relative to O, viz. the vector sum of α and β, might, so far as the final displacement of B is concerned, be made by giving B the *separate displacements α and β each relative to O*. For this reason B may be said to have the velocities $\bar{u}_1$, $\bar{u}_2$ relative to O *simultaneously*.

We may now define as follows:

DEFINITION. *A point is said to have* **two simultaneous independent velocities** *when its displacement in any time is the vector sum of the displacements which would be due to each velocity separately.*

The actual velocity of the point is (in accordance with the definition of § 10) called the resultant of the two independent velocities, and these latter are called the components.

A similar definition will apply for any number of independent velocities.

We may now succinctly state the proposition known as the Polygon of Velocities, given in § 28, as follows:

The resultant of any number of independent velocities is their vector sum.

Since an instantaneous velocity is equivalent to a certain constant velocity, and (§ 27) is subject to the same rule of composition as that constant velocity, all the above definitions of independent, component, and resultant velocities apply to instantaneous velocities.

The velocities are not necessarily all parallel to one plane.

30. On the Composition of Motions in general. Let a point P have any number of independent motions whatever; denote these by (A), (B), (C), ...

Let α be the displacement which would be due to the motion (A) alone in time t.

Let β be the displacement which would be due to the motion (B) alone in time t.

Let γ be the displacement which would be due to the motion (C) alone in time t, etc.

The displacement of P when all the motions go on simultaneously is the vector sum of α, β, γ, ...

To prove this we have only to remark that the resultant velocity at any instant is the vector sum of the velocities which would be due to each motion separately (§ 29).

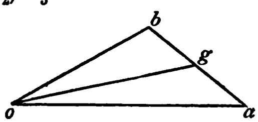
$\therefore$ the displacement which would be due to the resultant velocity in a small interval τ beginning at the instant is the vector sum of the displacements which would be due to the several independent velocities in the same interval.

The same is true for each small interval. By the principle of § 26, the final actual displacement can be made to approach as nearly as we please to the vector sum of these small displacements for all the intervals by sufficiently diminishing τ. But the set of displacements due to (A) add up to α, and the set of displacements due to (B) add up to β, and so on. Hence the proposition.

31. Calculation of the resultant of any number of independent velocities.

As we have seen, the resultant of any number of independent velocities may be found graphically by constructing their vector sum.

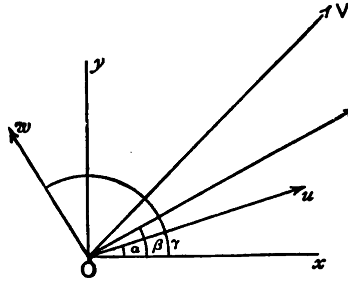
The following case deserves special notice: Let the velocities be represented by multiples, $m_1 oa$, $m_2 ob$, $m_3 oc$ ..., of straight lines oa, ob, oc ... drawn from a given point o, where m_1, m_2, m_3... are any scalars. Then (§ 16) the resultant is represented by $(m_1+m_2+m_3 \dots)\ og$, where g is the mean point of $a, b, c \dots$ for the multiples $m_1, m_2, m_3 \dots$.



In particular, the resultant of two velocities represented by oa, ob is $2og$, where g is the middle point of ab.

Cartesian Formulae for the Resultant.

To find the resultant of any number of coplanar velocities of magnitudes u, v, w ... inclined at angles α, β, γ ... to a given direction Ox. Let V be the magnitude of the resultant, and let the resultant be inclined at an angle ϕ to Ox.



Then since (§ 15) the resolved part of the vector sum = the algebraic sum of the resolved parts, taking resolved parts parallel to Ox we have

$$V\cos\phi = u\cos\alpha + v\cos\beta + w\cos\gamma + \ldots = \Sigma u\cos\alpha.$$

And similarly, taking the resolved parts *perpendicular* to Ox,

$$V\sin\phi = u\sin\alpha + v\sin\beta + w\sin\gamma + \ldots = \Sigma u\sin\alpha.$$

And therefore $V^2 = u^2 + v^2 + w^2 + \ldots + 2uv\cos(\alpha - \beta) + \ldots$ and $\tan\phi = \dfrac{\Sigma u\sin\alpha}{\Sigma u\cos\alpha}$.

Corollary. If there are only *two* velocities inclined at an angle θ, the resultant is given by $V^2 = u^2 + v^2 + 2uv\cos\theta$, and the inclination of the resultant to the velocity u by

$$\tan\phi = \frac{v\sin\alpha}{u + v\cos\alpha},$$

as may be easily seen independently.

The resultant of velocities not all parallel to a plane can be found in a similar manner by reference to *three* axes. The magnitude V of the resultant is given by

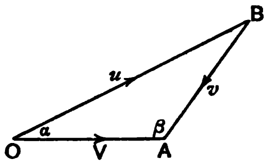
$$V^2 = (\Sigma u\cos\alpha_x)^2 + (\Sigma u\cos\alpha_y)^2 + (\Sigma u\cos\alpha_z)^2,$$

where α_x, α_y, α_z, are the inclinations to the axes of the velocity whose magnitude is u. The inclinations of the resultant to the axes are

$$\cos^{-1}\frac{\Sigma u\cos\alpha_x}{V},\quad \cos^{-1}\frac{\Sigma u\cos\alpha_y}{V},\quad \cos^{-1}\frac{\Sigma u\cos\alpha_z}{V}.$$

32. Resolution of a velocity into components.

I. *In Two Dimensions.* Let OA, numerically equal to V, represent the velocity. It may be resolved into two components represented by OB, BA in an infinite number of ways, because B may be chosen to be any point whatever. These components may be made to satisfy various conditions of which the following is a summary:



(1) Given the magnitudes, u, v, of the components, the direction of each is determinate. The resolution is possible if $u+v > V$.

(2) Given the directions, the magnitudes are determinate.

Let α, β be the angles the known directions make with the direction of V.

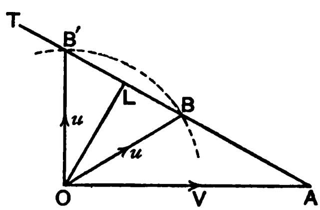
Then angle $OBA = \pi - (\alpha+\beta)$.

But by trigonometry, $\dfrac{OB}{\sin\beta} = \dfrac{BA}{\sin\alpha} = \dfrac{OA}{\sin(\alpha+\beta)}$;

therefore $u = \dfrac{V\sin\beta}{\sin(\alpha+\beta)}$, $v = \dfrac{V\sin\alpha}{\sin(\alpha+\beta)}$.

(3) Given the magnitude (u) of one component, and its direction, the other component is determinate.

(4) Given the magnitude (u) of one component and the direction of the other.



TA be the given direction.

Draw OL perpendicular to AT.

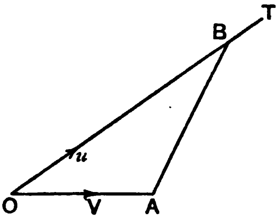
Then (i.) if $u < OL$, the resolution is impossible;

(ii.) if $u = OL$ there is one possible pair of components, and these are at right angles;

(iii.) if $u > OL$ but $< OA$ (V), there are two possible pairs of components;

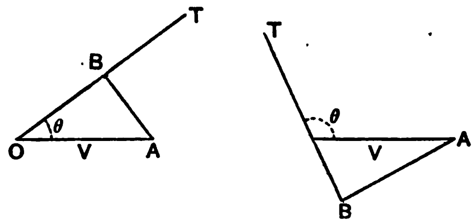
(iv.) if $u > OA$ there are also two pairs; but one of the v components is in the sense TA, the other in the sense AT.

(5) If the direction only of one component (u) is given, there are an infinite number of pairs of components.



Let OT be the direction of u. Take any point B on OT, then OB, BA represent a possible pair of components.

The case in which the pair of components is at right angles is of supreme importance. The components OB, BA are then the



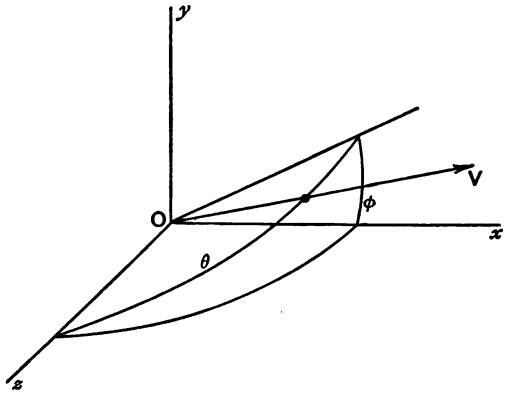
resolved parts (see § 15 above) of V parallel and perpendicular to the given direction.

Their magnitudes are therefore $V \cos \theta$ parallel to the given direction, $V \cos (90° - \theta)$ or $V \sin \theta$ perpendicular to it, θ being the angle (acute or obtuse) between the directions of u and V. When θ is obtuse, the u component is in the sense TO.

II. *In Three Dimensions.* An important case of resolution into three components mutually at right angles is the following:

Let the velocity, of magnitude V, be inclined to the axis of z at an angle θ, and let the plane zOV be inclined to the plane zOx at an angle ϕ.

Then the resolved parts of V are $V \sin\theta\cos\phi$ parallel to Ox,



$V \sin\theta \sin\phi$ parallel to Oy, $V\cos\theta$ parallel to Oz.

33. Velocity of the Mean Point (centroid). Let α, β, γ, ... be the position vectors of the points A, B, C ... relative to any origin O. Then the position vector of G the mean point of A, B, C, ... for the multiples m_1, m_2, m_3 is

$$\frac{1}{\Sigma m}\{m_1\alpha + m_2\beta + m_3\gamma + \ldots\},$$

m_1, m_2, m_3, ... being any scalars.

Now, the velocity of a point whose position vector is ρ being denoted by $\dot{\rho}$, that of a point whose position vector is $m\rho$ (m being a scalar), is $m\dot{\rho}$; for when ρ becomes $\rho+\delta$, $m\rho$ becomes $m(\rho+\delta)$; the displacements are thus in the ratio $1:m$; so consequently are the average velocities for a given interval, and so therefore are the velocities at a given instant.

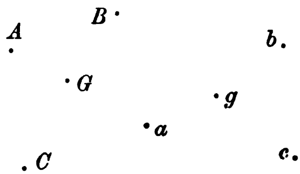
Hence, by the polygon of velocities, § 28, the velocity of G relative to O is

$$\frac{m_1}{\Sigma m}\cdot\dot{\alpha} + \frac{m_2}{\Sigma m}\cdot\dot{\beta} + \ldots, \text{ or } \frac{\Sigma m\dot{\alpha}}{\Sigma m}.$$

Corollary. If the resolved parts of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, ... parallel to any line be of magnitudes u_1, u_2, u_3, ... the resolved part of the velocity of G parallel to this line is of magnitude $\frac{\Sigma mu}{\Sigma m}$.

This result evidently appears in the *Velocity Diagram* as follows:

Let A, B, C, ... be the points whose velocities we are considering, and a, b, c, ... the points in the velocity diagram corresponding to A, B, C, ... respectively.



Let G be the mean point of A, B, C, ... for the multiples $m_1, m_2, m_3, \ldots$. And let g be the mean point of a, b, c, ... for the same multiples. Then the velocity of G relative to *any* point O in the diagram A, B, C, ... is represented by og, where o is the point corresponding to O in the velocity diagram.

Examples.

1. A point has four simultaneous velocities of 20 cm.s. The angle between the first and second is 30°, that between the second and third is 60°, and that between the third and fourth is 30°.

   Find the magnitude and direction of their resultant, (i) by a graphical construction, using mathematical instruments; (ii) by calculation.

2. A ship is sailing due south at the rate of 4 f.s.; a current is carrying it due east at the rate of 3 f.s., and a sailor is climbing up a vertical mast at the rate of 2 f.s. What are the speeds of the ship and the man?

3. A point has independent velocities 1, $-\frac{1}{\underline{|2}}$, $+\frac{1}{\underline{|4}}$, $-\frac{1}{\underline{|6}}$, etc., parallel to a given line, and 1, $-\frac{1}{\underline{|3}}$, $+\frac{1}{\underline{|5}}$, etc., perpendicular to this line.

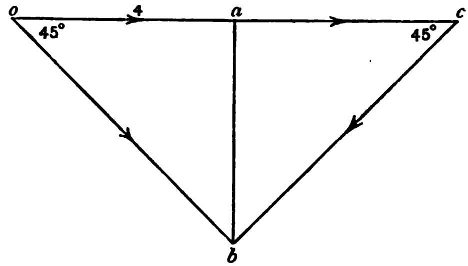
   Verify with mathematical instruments that its resultant velocity is unity, inclined at an angle of 1 radian to the given line.

4. Assuming the law of composition of velocities and some rule for drawing a tangent to a parabola, find the velocity at any instant of a point moving with the horizontal component of its motion uniform in a parabola whose axis is vertical.

5. A person moving due east at 4 miles an hour observes that the wind seems to blow directly from the north. On doubling his speed, it appears to blow from the north-east; find the true velocity of the wind.

[Draw the velocity diagram. Let oa represent the person's first, oc his second velocity, so that $oa = ac$. He observes the velocity of

the wind relative to himself. Drawing ab from N. to S., cb from N.E. to S.W., these are the directions of the relative velocities.



Then joining ob, ob is the actual velocity of the wind. Angle $acb=45°$, $ba=ac=oa$; $\therefore\ ob=4\sqrt{2}$ miles an hour, and angle $boa=45°$, or the wind is in the N.W.]

6. Two points are moving with constant velocities in straight lines AB and CD, and in the same second they move from A to B and C to D respectively. Find by geometrical construction the position in which they are nearest together.

7. AB, CD are two equal straight lines bisecting each other at right angles. Points start at the same instant from A, C and move along AB, CD with constant velocities. Show that, the relative velocity of the points being given in magnitude, the time that elapses before they are at their shortest distance apart is proportional to the sum of their speeds.

8. Show that the trail of smoke from the funnel of a steamer moving with constant velocity is in a vertical plane parallel to the vane on the mast.

9. A steamer is going down a straight river at constant speed, and its flags make an angle α with the direction of motion. On turning through a right angle the flags make an angle β with the direction of motion. Find the direction of the wind.

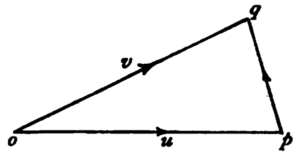
10. OA, OB, OC are three straight lines meeting in O. Angle $AOB=30°$, angle $BOC=60°$. Three points P, Q, R start simultaneously from O and travel along OA, OB, OC respectively with constant velocities such that P, Q, R are always in a straight line perpendicular to OB. Prove that the velocities of Q and R relative to P are in the ratio $1:4$.

11. Two points move along straight lines at right angles to one another, each with a speed of 7 f.s. Determine their relative velocity, and show that if one of them passes through the point of intersection of their paths a second later than the other, the time between the two occasions at which their distance apart is 5 ft. will be $\frac{1}{7}$ of a second.

34. Change of Velocity.

Definition. *If a point has at any instant a velocity* u, *and after any interval a velocity* v, *the change of velocity during the interval is that velocity which compounded with* u *gives* v.

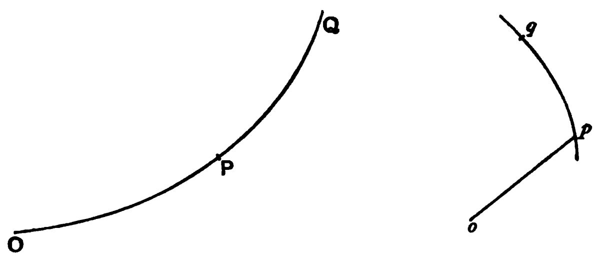
It is evident from § 29 that the *change of velocity* is the vector difference of $\bar{u}$ from $\bar{v}$. It is to be noted that the *speed* may be the same throughout the interval, and yet there may be a change of velocity.



Thus, let op, oq represent the velocities u, v, and suppose that op, oq are equal in magnitude; the vector pq is still in general finite.

Hodograph.

Let a point P be describing a curve, and from a fixed point o let there be drawn op to represent the velocity of P, so that as P moves along its path, op turns so as to be always parallel to



the tangent at P, and grows or diminishes so that the number of units in its length is always numerically equal to the speed of P; p traces out a new curve called the *hodograph*.

Any radius to the new curve from o represents the velocity at the corresponding point of the path. Hence the name (*hodon graphei*—it describes the way). The point o is called the pole of the hodograph.

If P, Q be two points on the path, p, q the corresponding points of the hodograph, pq the chord of the hodograph represents the change of velocity as the moving point travels from P to Q. The *arc* pq of the hodograph may be regarded as the limit of a polygon whose sides represent successive small changes of velocity whose vector sum is the chord pq.

Examples.

1. A point moves with speed 10 cm.s. along the base BC of an equilateral triangle; on arriving at C it proceeds with double speed along the next side. Find the change of velocity.

2. A point proceeds with constant speed v round the perimeter of a regular octagon. Prove that the change of velocity at each angular point is equal to $v\sqrt{2-\sqrt{2}}$, along the radius of the circum-circle of the octagon.

3. A point travels with constant speed v from A to B along the circumference of a circle, centre O. If the angle AOB be θ, prove that the change of velocity as the point goes from A to B is $2v \sin\frac{\theta}{2}$ in a direction parallel to the bisector of the angle AOB.

4. If the moving point P travels with constant velocity, the hodograph is a point; if with velocity constant in direction only, the hodograph is a straight line through the pole parallel to this direction.

5. If the point P describes the successive sides of a regular hexagon with constant speed, what is the hodograph?

6. A point moves in a circle, and its speed is proportional to its distance from a fixed diameter of the circle; find the hodograph.

7. A point P moves in an ellipse in such a way that if its velocity be resolved into two components parallel to HP, PS the focal distances, these components are constant in magnitude. Find the hodograph.

35. Acceleration.

DEFINITION. *Acceleration is the time-rate of change of velocity.*

On Constant Acceleration.

DEFINITION. *The acceleration of a point is constant when equal changes of velocity take place in* **any two** *equal times.*

The phrase "equal changes" implies, of course, that the changes of velocity must be in the same direction.

Constant acceleration is a vector quantity.

Its magnitude = the magnitude of the change of velocity in one second.

The direction of the acceleration is the direction of the change of velocity.

The Unit of Acceleration is the acceleration of a point which receives unit change of velocity in unit time.

The unit of acceleration is usually spoken of as "a foot (or centimetre) per second per second."

These units are for brevity denoted by 1 f.s.s. and 1 cm.s.s. respectively.

The rate of change of *speed* may be called speed-acceleration. It is a scalar quantity, and is not in general equal in magnitude to the *acceleration*. (See forward, § 49.)

The double reference to *time* in the measure of acceleration is clearly brought out in the following examples:

1. A point moves from rest in such a manner that at the end of a minute it has acquired a velocity of 3600 feet per minute. What is its acceleration?

A velocity of 3600 feet per minute $= \frac{3600}{60}$ feet per second $= 60$ units of velocity.

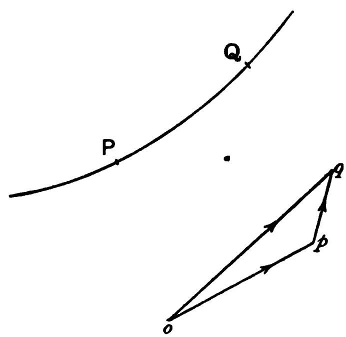
$\therefore$ 60 units of velocity are acquired in a minute.

$\therefore$ $\frac{60}{60}$, or 1 unit of velocity, is acquired in a second.

$\therefore$ the acceleration is unity, or 1 foot per second per second.

2. The measure of an acceleration in the C.G.S. system is 981; what is its measure when a yard and a minute are units, if 1 foot be taken equal to 30·5 cm.?

36. Variable Acceleration. Let a point P be describing any path. Suppose the point describes the arc PQ in the interval τ. Let op, oq represent the instantaneous velocities at P and Q. pq represents the change of velocity in the interval τ; $\left(\frac{pq}{\tau}\right)$ represents the *average acceleration* for that interval, which



is defined as *that constant acceleration which will produce the actual change of velocity in the interval* τ.

Acceleration at any instant of the motion.

DEFINITION. *If the average acceleration for an interval τ including the particular instant can be made to approach as nearly as we please to a definite vector by diminishing τ indefinitely, the point is said to have an acceleration* at *the instant and its value is the limit to which the average acceleration* $\left(\frac{\text{pq}}{\tau}\right)$ *tends.*

As in the case of instantaneous velocity, unless the same limit is obtained for all positions of the interval with regard to the instant, the point cannot be said to have an acceleration at the instant.

From the definition we see that the instantaneous acceleration of P is represented by the instantaneous velocity of p in the hodograph. This is a most important result: it enables us at once without further proof **to transform any velocity theorem into an acceleration theorem.**

Thus the theorem of § 26 becomes: If a point has an acceleration at each instant of its motion, its velocity at any instant may be found to any required degree of accuracy by supposing the point to travel for a short interval τ with the acceleration it has at the beginning of this interval, for another short interval τ with the acceleration it has at the beginning of this other interval, and so on.

The student acquainted with the elements of the differential calculus will note that if v be the *speed*, the speed-acceleration is equal to $\frac{dv}{dt}$. In Newton's notation we may denote it by $\dot{v}$, a scalar symbol. If $\dot{\rho}$ represent the *velocity*, $\ddot{\rho}$, a vector symbol, is used to denote the acceleration.

Examples.

1. A point moves in a curved path with constant acceleration; prove that v the speed at any point of the path is proportional to $\operatorname{cosec}\phi$, where ϕ is the angle between the direction of the velocity at that point and that of the acceleration.

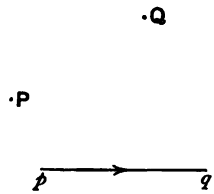
[Here the hodograph is a straight line. See § 40 below.]

2. If the direction of the acceleration of a point describing a curve always bisects the angle between the tangent and normal, find the velocity at any point in terms of the angle which the normal at the point makes with a fixed line.

[The hodograph is an equiangular spiral; if v_0 be the speed when the normal is parallel to the fixed line, v the speed when the inclination of the normal to the fixed line is θ, $v=v_0e^{\theta}$.]

3. In what time will a constant acceleration f change a velocity of magnitude v into a velocity of magnitude v' in a direction making an angle θ with the direction of the former velocity?

37. Acceleration, like velocity, is always relative. The acceleration of a point Q relative to a point P is the rate of change of Q's velocity relative to P. If the vector pq represent this relative velocity, the acceleration of Q relative to P is represented by the rate of change of this vector, that is, by the velocity of q relative to p.

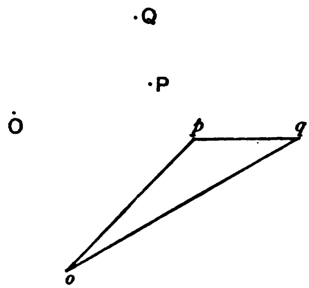


38. Parallelogram of Accelerations.

PROPOSITION. *If a point* Q *has an acceleration* $\bar{f}_2$ *relative to a point* P, *and* P *an acceleration* $\bar{f}_1$ *relative to a point* O, *the acceleration of* Q *relative to* O *is the vector sum of* $\bar{f}_1$ *and* $\bar{f}_2$.

[This at once follows, as remarked in § 36, from the corresponding velocity theorem, but the student may find it convenient to have a formal proof.]

Let op represent the velocity of P relative to O, pq that of Q relative to P. Therefore oq represents that of Q relative to O. Now the velocity of q relative to o is the vector sum of the velocity of q relative to p and that of p relative to o.



But the velocity of

q	relative to	p	is the acceleration of	Q	relative to	P.
p	„	o	„	P	„	O.
q	„	o	„	Q	„	O.

which proves the proposition.

Corollary. If $A, B, C, \ldots$ be any number of moving points, and if $\bar{f}_1$ be the acceleration of A relative to O, $\bar{f}_2$ that of B relative to A, $\bar{f}_3$ that of C relative to B, and so on, the acceleration of the last point (K) of the series relative to O is the vector sum of $\bar{f}_1, \bar{f}_2, \bar{f}_3, \ldots$.

Thus, the position vector of K relative to O being the vector sum $\alpha+\beta+\gamma+\dots$, and K's velocity $\dot{\alpha}+\dot{\beta}+\dot{\gamma}+\dots$, its acceleration may be written $\ddot{\alpha}+\ddot{\beta}+\ddot{\gamma}+\dots$.

39. On Component Accelerations. If in the last article we interchange the accelerations of P and Q, giving

Q the acceleration $\bar{f}_1$ relative to P, and

P „ $\bar{f}_2$ „ O,

just as before the acceleration of Q relative to O would be the vector sum of $\bar{f}_1$ and $\bar{f}_2$, the order in which the vector sum is taken making no difference to the result.

We express this by saying (as in § 29) that the point Q has two "independent" accelerations relative to O, and we may define as follows: A point is said to have two or more simultaneous independent accelerations when its change of velocity in any time is the vector sum of the changes of velocity which would be due to each acceleration separately. The actual acceleration of the point is called the resultant of the two independent accelerations, and these latter are called the components. Similar definitions hold good for any number of independent accelerations, and we may now state the result of the last paragraph in the following form:

The resultant of any number of independent accelerations is their vector sum.

All the geometrical results of §§ 31–33 hold good for the composition of accelerations.

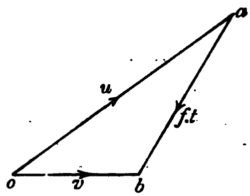
For example, from § 33, replacing position vectors by velocities and velocities by accelerations, we see that if $\alpha, \beta, \gamma, \dots$ be the position vectors of $A, B, C\dots$ relative to a given origin O, the acceleration relative to O of the mean point of $A, B, C, \dots$ for the multiples $m_1, m_2, m_3, \dots$ is $\frac{\Sigma m\ddot{\alpha}}{\Sigma m}$, or, if $f_1, f_2, \dots$ be the resolved accelerations of $A, B, C, \dots$ parallel to Ox, the resolved acceleration of the mean point is $\frac{\Sigma mf}{\Sigma m}$.

40. Motion under Constant Acceleration.

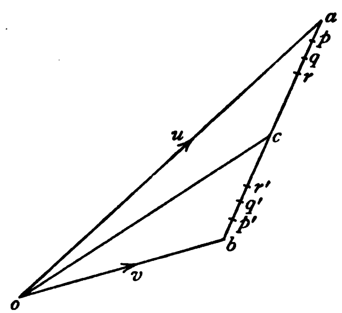
(i.) *If a point has an initial velocity* u *and a constant acceleration* f, *its velocity at the end of* t *seconds is the vector sum* $\bar{\text{u}}+\bar{\text{f}}\text{t}$.

Since the acceleration is constant, the hodograph is a straight line described with speed f.

Draw oa to represent the initial velocity u, and ab parallel to the direction of f. Then ab is the hodograph. Make $ab = ft$; then b is the position of the point tracing the hodograph at the end of t seconds. The velocity in the path is then ob, or $\bar{u} + \bar{f}t$.



(ii.) *Under the same circumstances the displacement of the moving point in time* t *is* $\bar{u}t + \frac{1}{2}\bar{f}t^2$.



oa representing the initial, ob the final velocity, bisect ab at c. Then $ac = cb = f \cdot \frac{t}{2}$, and oc is the velocity at half-time.

Divide $\frac{t}{2}$ into n intervals each equal to τ, and let $ap = pq = qr = \ldots = r'q' = q'p' = p'b = f \cdot \tau$.

To get the actual displacement, we may by the theorem of § 26 suppose the point to move for successive intervals each equal to τ with velocities oa, op, oq, ... the velocities at the *beginnings of the intervals*, and then after half-time with velocities ... oq', op', ob, *the velocities* at the *ends of the intervals*, provided we diminish τ indefinitely. Now the pairs of velocities oa, ob; op, op'; oq, oq'; ... are each equivalent to twice the velocity oc (§ 31). Therefore the vector sum of all these velocities $= 2n \cdot oc$.

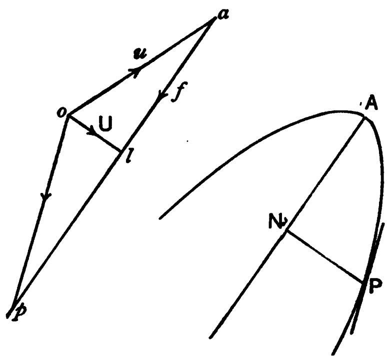
Hence the vector sum of the displacements $oa \cdot \tau$, $op \cdot \tau$, etc. ...

$$= 2n \cdot oc \cdot \tau = oc \cdot t = (\bar{u} + \tfrac{1}{2}\bar{f}t)t$$
$$= \bar{u}t + \tfrac{1}{2}\bar{f}t^2.$$

Hence, diminishing τ indefinitely, the actual displacement in time $t = \bar{u}t + \frac{1}{2}\bar{f}t^2$.

Corollary. The velocity oc is the velocity which would produce the *actual* displacement in time t. It is therefore the *average* velocity for this interval. Thus in motion under constant acceleration the average velocity for any interval of time is the velocity at the middle of the interval; it is also half the vector sum of the initial and final velocities.

(iii.) *Under the same circumstances, to find the path of the point.*



Draw ol perpendicular to the hodograph.

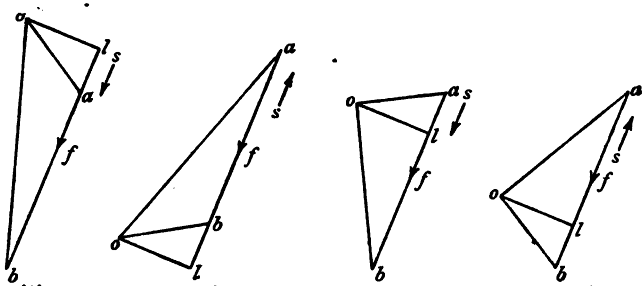
Let A be the point of the path corresponding to the velocity ol, and let $ol = U$.

Then the position of the point at any subsequent time t is obtained by drawing the displacements $AN = \frac{1}{2}ft^2$ parallel to al, and $NP = U \cdot t$ parallel to ol.

Hence $PN^2 = \dfrac{2U^2}{f} \cdot AN$, or the path is a parabola whose axis is parallel to the direction of the acceleration and whose latus rectum $= \dfrac{2U^2}{f}$, where U is the constant component of the velocity perpendicular to this direction.

If the point does not actually pass through A but starts beyond it, we can still suppose that the velocity at P was generated by starting the point from A, under the given acceleration, and the above reasoning is still applicable.

(iv.) *If* u, v *be the initial and final speeds, and* s *be the resolved displacement in the direction of the acceleration,* s *being reckoned positive when it is in the same sense as* f, *these quantities are connected by the scalar equation* $v^2 = u^2 + 2fs$.



s positive *s negative* *s positive* *s negative*

ol being perpendicular to ab or ab produced, it follows that

$$\begin{aligned} v^2 - u^2 &= ob^2 - oa^2 \\ &= lb^2 - la^2 \\ &= (lb + la)(lb - la). \end{aligned}$$

Now $\dfrac{lb + la}{2}$ = speed of the average velocity resolved in the direction of the acceleration, $= + \dfrac{s}{t}$, the positive sign being taken in all cases if s is reckoned positive in the sense of f, negative in the sense opposite to f.

Also $\quad lb - la = ab = ft$, in all cases.

$$\therefore \quad v^2 - u^2 = \frac{2s}{t} \cdot ft = 2fs,$$

or,

$$v^2 = u^2 + 2fs.$$

41. Motion under constant acceleration when the direction of the initial velocity u coincides with that of the acceleration f.

In this case the vector sums of § 40 become algebraic, and we

have, u denoting the initial, v the final velocity, f the acceleration, s the displacement,

$$v = u + ft, \text{......................from (i.)}$$
$$s = ut + \tfrac{1}{2}ft^2, \text{....................from (ii.)}$$
$$v^2 = u^2 + 2fs. \text{....................from (iv.)}$$

If the sense of f is opposite to that of u, the acceleration is a *retardation*; u will then be negative in the first two of these formulae, and v and s will at first also be negative, becoming positive, however, after a sufficient time has elapsed.

Examples.

1. In the case of rectilinear motion, deduce the third formula from the other two.

[Square the first; $\therefore\ v^2 = u^2 + 2ft \,.\, u + f^2t^2$
$= u^2 + 2f(ut + \tfrac{1}{2}ft^2)$
$= u^2 + 2fs.$]

2. If a point be moving in a straight line with constant retardation, it will pass twice through any point in part of its path with velocities equal in magnitude and opposite in direction. What is the corresponding theorem when the motion is not rectilinear, but still under constant acceleration?

42. Vertical Motion under Gravity. Experiment shows that if a body be let fall towards the earth, or be projected vertically upwards from the earth's surface, the motion taking place in a vacuum, the body moves with an acceleration (or retardation) which is the same for all bodies at the same spot, but is slightly different at different spots, on the earth's surface. This acceleration is, to a close approximation, constant during the body's motion, its direction being the vertical, defined as the line in which a plumb-line hangs; this acceleration is called the acceleration due to gravity; its numerical value at a given spot on the earth's surface is determined not by direct experiments on falling bodies but by observations on pendulums; some account of the essential points of such an observation is given in Chapter VIII. § 170. This numerical value is denoted by g. The value of g in the latitude of London is 32·19 f.s.s., or 981 cm.s.s., approximately. At any point on the earth's surface whose latitude is λ its value is given by Clairaut's Formula, viz.:

$$g = g_0\,(1 + \cdot 005133 \sin^2\lambda),$$

where g_0 is the value of g at the equator.

43. In discussing examples of vertical motion under gravity, we shall find it convenient in all cases to adopt the convention of sign employed in § 40 (iv.). The positive sense for velocities and displacements will thus be *downwards* in all cases. If, then, u and h be positive numbers representing numerically the initial velocity of a body thrown upwards and its height at time t, we shall have

$$v = -u + gt \ldots (1), \quad -h = -ut + \tfrac{1}{2}gt^2 \ldots\ldots (2)$$

and
$$v^2 = u^2 - 2gh. \ldots\ldots (3)$$

It is to be noted that equation (2) gives only the actual displacement of the body from its starting-point at time t, and *not* the length of path traversed in time t, the two being identical only on the journey *away* from the starting-point.

The following elementary propositions should be noticed. (1) When a body is projected vertically upwards, the speed at height h is given by $v = \pm\sqrt{u^2 - 2gh}$, the positive sign corresponding to a velocity *downwards*; therefore the body always passes a given point of its path with equal speeds on the upward and downward journeys. (2) Hence, by the help of the proposition of § 26, it appears that any portion of the path is described in equal times on the upward and downward journeys. (3) The velocity due to a fall from a height h is given by $v = \sqrt{2gh}$. This is often called "the velocity due to a height h."

In the following examples the resistance of the air is to be neglected.

Examples.

1. Taking 1 cm. = ·3937 inches, verify that an acceleration of 32·19 f.s.s. is equivalent to an acceleration of 981 cm.s.s.

2. A stone is thrown vertically upwards with a speed of 1000 cm.s. from the extreme edge of a cliff 300 metres high. After what time will it reach the base of the cliff?

[Taking the formula $s = ut + \frac{1}{2}ft^2$, we must, with the above convention with regard to signs, write

$$u = -1000, \quad f = g = +981, \quad s = +30{,}000,$$

obtaining the quadratic $981t^2 - 2000t - 60{,}000 = 0$.

The positive root gives 8·9 seconds approximately for the required time.]

3. Interpret the negative root of the quadratic in Question 2.

4. A stone falls vertically from rest for 3 seconds. Another is then thrown down after it with a velocity of $5g$ f.s. When and where will they meet? [$g = 32$ f.s.s.]

[Use relative velocities.

The relative velocity of the two when the second starts is $5g - 3g$, or $2g$ f.s., and the two are then $\frac{1}{2}g(3)^2$ or $\frac{9g}{2}$ feet apart.

The acceleration of both is the same, and therefore the relative velocity remains the same throughout the motion.

$\therefore$ the second catches the first up after $\frac{9g}{2} \div 2g$, or $2\frac{1}{4}$ seconds.

The first stone has now been moving for $5\frac{1}{4}$ seconds, and is therefore at a depth of $\frac{1}{2}g(5\frac{1}{4})^2$, or 441 feet.]

5. Two balls are projected at the same instant, one downwards with a speed of 60 f.s. from the top of a tower 55 ft. high, the other upwards from the bottom with a speed of 50 f.s. When and where will they meet? [$g = 32$ f.s.s.]

6. A stone is thrown vertically upwards with a speed of 50 f.s., a second later another stone is thrown vertically upwards with a speed of 40 f.s.; taking g as equal to 32 f.s.s., find when and where the stones will meet.

What is the *least* speed with which the second stone can be projected in order that they may meet while both are still in motion?

7. A point having constant acceleration is observed to pass over 100 feet in a certain second, and after an interval of 3 seconds is observed to pass over 148 feet in the next second. When was it at rest?

8. A point is moving in a straight line with constant acceleration. If the speeds at four points be in arithmetical progression, the distances between those points are also in arithmetical progression.

9. Two points start simultaneously from the same point to traverse a closed path, 144 feet long, in opposite directions: the one moves with a constant speed of 7 f.s., the other, whose initial speed is zero, has a constant speed-acceleration of 2 f.s.s. When and where do they next meet? Account for the negative solution.

10. A number of points start simultaneously from a given point O with equal speeds in different directions but the same constant acceleration. Prove that at any subsequent instant they all lie on a sphere whose centre is distant $\frac{1}{2}ft^2$ from O, f being the numerical value of the acceleration.

11. A train stopping at two stations 2 miles apart on a straight line takes 4 minutes for the journey. Assuming that its motion is first uniformly accelerated and then uniformly retarded, prove that $\frac{1}{x} + \frac{1}{y} = 4$, where x and y are the magnitudes of the acceleration and retardation respectively, a mile and a minute being the units.

12. A number of points start simultaneously from a given point O with different velocities and the same constant acceleration;

prove that at any subsequent instant their directions of motion all meet in a point.

13. Two points A, B are moving in a plane, each with constant acceleration. Prove that the path of either relative to the other is a parabola, and find its latus rectum.

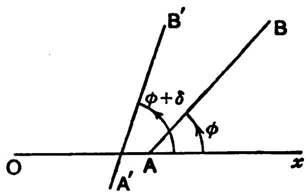
44. Angular Velocity and Angular Acceleration.

Definition. *The angular velocity of a straight line* AB *relative to another straight line* CD, *both the lines being in one plane, is the rate of change of the angle between their positive directions.*

As a rule the angular velocities of all lines in the plane are estimated relative to some specified line. The phrase "angular velocity" without qualification means angular velocity relative to this line.

Angular velocity measures the rate of turning only, and is quite independent of any motion of translation the straight line may have.

Let Ox be the specified line, and let the positive sense of the straight line AB make an angle ϕ with Ox. Suppose that AB moves to $A'B'$ in the small interval τ, thus changing ϕ into $\phi+\delta$.



Then the angular velocity of AB at any instant is the limit (if any) to which $\frac{\delta}{\tau}$ tends as τ is diminished indefinitely, the interval τ being taken indifferently so that the instant coincides with either end of it, or falls between them. (See § 24. Definition of variable velocity).

The angular velocity is reckoned positive if AB is rotating counterclockwise from Ox, otherwise negative.

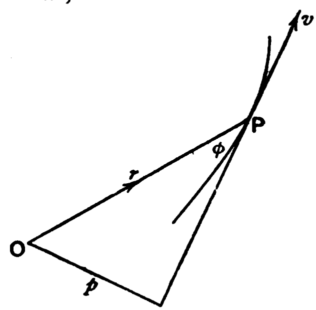
The measure of an angular velocity is usually denoted by ω, or in Newton's Notation, by $\dot{\theta}$, $\dot{\phi}$, etc.

Relative angular velocities in one plane may clearly be compounded by algebraic addition.*

The unit of angular velocity usually adopted is an angular velocity of one radian per second.

* An angular velocity is really a vector; it may be represented by a straight line of given direction and sense drawn at right angles to its plane; the algebraic addition of the text is a particular case of vector addition.

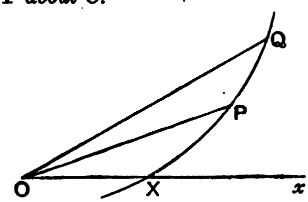
tangent to P's path; thus $v \sin \phi = \frac{vp}{r}$ = component of v perpendicular to $OP = \omega r$;



$$\therefore \quad \omega = \frac{v \sin \phi}{r}, \text{ or } \omega = \frac{pv}{r^2}. \quad \text{Also clearly } \dot{r} = v \cos \phi.$$

47. Areal Velocity.

DEFINITION. *If the point* P *is moving along any line* XP *which cuts* Ox *in* X, *the rate of increase of the area* XOP *is called the areal velocity of* P *about* O.



Let P, Q be the positions of the moving point at the beginning and end of a small interval τ, and let angle $POQ = \delta$.

Then area $POQ = \frac{1}{2} OP \,.\, OQ \sin \delta$.

And the areal velocity of P about O

$$= \text{the limit of } \tfrac{1}{2} \,.\, OP \,.\, OQ \,.\, \frac{\sin \delta}{\delta} \cdot \frac{\delta}{\tau},$$

when τ is indefinitely diminished,

or the areal velocity $= \frac{1}{2} r^2 \omega = \frac{1}{2} vp$,

where r, ω, v, p have the same meanings as in the preceding section.

Rate of change of areal velocity is called areal acceleration.

Examples.

1. Assuming that the moon travels in a circle round the earth's centre once in $27\frac{1}{3}$ days, the plane of the moon's equator being the same as that of the circle, and that the point of the moon's surface nearest the centre of the earth is always the same point, find the angular velocity of any point on the moon's equator about the moon's centre.

2. A point moves with constant speed in a straight line. Prove that its angular velocity about a given point varies inversely as the square of the distance from that point.

3. A point moves with constant speed in a parabola. Prove that its angular velocity about the focus is proportional to $\dfrac{1}{SP^{\frac{3}{2}}}$.

4. A point moves with constant speed in an ellipse. Prove that its *areal* velocity about the centre is inversely proportional to the diameter conjugate to that through the point. Also if the areal velocity is constant the speed is proportional to the conjugate diameter.

5. Find the velocity of an insect that crawls at a rate of 10 feet per minute along a spoke of a flywheel that makes 5 revolutions per minute, when it is at a distance of 2 feet from the centre of the wheel.

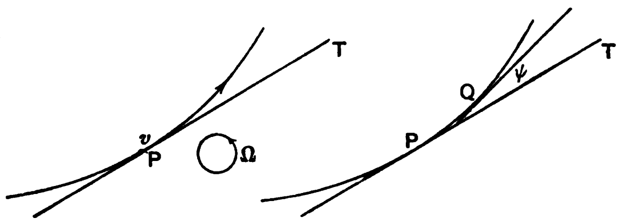
6. A straight line AB of invariable length is moving in a plane. Prove that if the velocities of A and B are parallel they are either equal or at right angles to AB.

7. A point describes a circle with constant angular velocity about a point in the circumference. What is the hodograph?

8. AB, BC, CD, ... are a number of equal laths, each of length a, pivoted together at B, C, D, ... $A'B'$, $B'C'$, $C'D'$, ... are an equal number of similar laths of length a, similarly pivoted. AB and $A'B'$ are pivoted together at their middle points in the form of an **X**, as are also BC and $B'C'$, CD and $C'D'$, ... If A, A' are moved towards each other with given equal speeds v, find the velocities of the various joints B, B', C, C' ... etc., as also the velocities of the middle points of the laths.

48. Curvature. Let P be a point describing a path in a plane with instantaneous speed v, and let a straight line be supposed to move in such a manner that as P describes the path the straight line is always a tangent to the path at P; this straight line will for any given position of P have an instantaneous angular velocity, which we may call the *angular velocity*

of the tangent. Denote this angular velocity by Ω. Then the ratio $\frac{\Omega}{v}$ is called the *curvature* of the path at P.



If PQ be an arc of the curve described, ψ the angle between the tangents at P and Q, the curvature at P is clearly the limit to which the ratio $\frac{\psi}{arc\ PQ}$ tends as Q is made to move up to P. It is thus independent of the particular value of v for which the moving point passes through P. Thus the ratio $\frac{\Omega}{v}$ is the same at the same point of a given path for all values of v.

When the path is a circle, the tangent is perpendicular to the radius at all points, and therefore has the same angular velocity as the radius. Hence, r being the radius, $v=r\Omega$, (§ 45) or $\frac{\Omega}{v}=\frac{1}{r}$. *Thus the curvature of a circle at any point is measured by the reciprocal of its radius.*

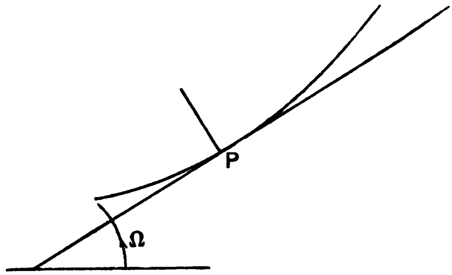
DEFINITION. *A circle which has the same curvature as a given curve at a given point is called the* **circle of curvature** *at that point. The radius of the circle is called the* **radius of curvature** *at that point.*

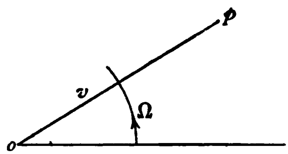
If the radius of curvature of a curve at a given point is R, the curvature at that point is evidently equal to $\frac{1}{R}$.

49. *A point* P *is moving in any manner in a plane. To find the resolved parts of its acceleration parallel and perpendicular to the tangent at* P *to the path.*

Let v be the *speed* of P, f the rate of change of v, *i.e.* the *speed-acceleration*, Ω the angular velocity of the tangent at P. Draw op to represent the velocity at P, so that p traces out the hodograph. Then $op=v$ numerically, and being always parallel

to the tangent is revolving instantaneously with angular velocity Ω. The acceleration of P is the velocity of p.





Now by § 46 the *velocity of p* parallel to *op* (*i.e.* to the tangent at P) is equal to the rate of change of the *length op*, that is to the *speed-acceleration f*.

Also by the same article the velocity of p perpendicular to *op* (*i.e.* to the tangent at P) is $op \,.\, \Omega$, or $v\Omega$ parallel to the inward normal at P. Let R be the radius of curvature of the path at P, then

$$\frac{\Omega}{v}=\frac{1}{R}, \text{ and } v\Omega=\frac{v^2}{R}=\Omega^2 R.$$

Thus the acceleration of P may be resolved into an acceleration parallel to the tangent, and numerically equal to f the speed-acceleration, and an acceleration $\frac{v^2}{R}$ $(=\Omega^2 R)$ inwards along the normal.

In Newton's notation f may be written $\ddot{s}$, and the normal acceleration $\frac{\dot{s}^2}{R}$ or $R\dot{\psi}^2$, s being the length of arc measured from a fixed point, ψ the angle made by the tangent with a fixed line.

Note that the resolved part of the acceleration along the tangent is *entirely* due to the *change of speed*, while that along the normal is *entirely* due to the *change of direction* of the velocity.

Corollary. If the path be a circle, radius r, described with constant speed v, the acceleration is *entirely along the normal* (inwards) and $=\frac{v^2}{r}$. The moving point is said then to have *Uniform Circular Motion.*

Further, if two points A, B are moving in a plane in such a manner that the length AB is constant, and that AB is moving with angular velocity ω, the resolved part of the acceleration of B relative to A in the direction BA is $\omega^2 . BA$. If ω is constant, this is the total acceleration of B relative to A.

Examples.

1. Prove that the resultant of two uniform circular motions in one plane of the same period and sense is a uniform circular motion.

2. Prove that a point moving in a plane with acceleration constant in magnitude and always at right angles to the velocity must be moving with uniform circular motion.

3. Prove that, if the *speed-acceleration* is at all points of the path equal in magnitude to the *acceleration*, the path of the moving point must be a straight line.

4. A point is describing a certain plane curve, and at a certain point on the curve its speed is found to be 2 f.s., while the angular velocity of the tangent is 3 radians per second; if the speed at this point were 3 f.s., what would the angular velocity of the tangent be?

5. Assuming that the moon revolves uniformly in a circle relative to the earth's centre, and that the acceleration of a body due to the gravity of the earth is inversely proportional to the square of the distance of the body from the earth's centre, prove that the acceleration of the moon relative to the earth's centre is very nearly that which would be due to the earth's gravitation.

[Earth's radius $=4000$ miles, radius of moon's orbit $=60$ times earth's radius, periodic time of moon in orbit $=27\frac{1}{3}$ days, value of g at earth's surface $=32{\cdot}2$.

The acceleration of the moon relative to the earth's centre as calculated from the orbit $= {\cdot}00897$ f.s.s.; that due to gravity at the distance of the moon $= {\cdot}00894$ f.s.s.]

6. Prove that the radius of curvature at a point P of an equiangular spiral, angle α, is r cosec α, where r is the distance of P from the pole.

7. A point P moves in a plane; if O be the origin, and if OP $(=r)$ increases with constant speed-acceleration $\ddot{r}$ while OP turns

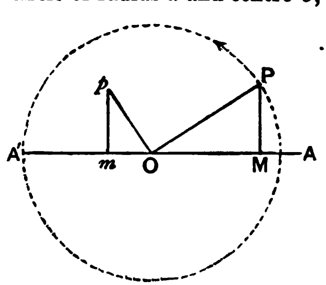
with constant angular velocity ω, prove (using the hodograph and the principle of relative velocities) that the accelerations of P parallel and perpendicular to OP are numerically equal to $\ddot{r}-r\omega^2$ and $2\dot{r}\omega$.

50. Simple Harmonic Motion.

DEFINITION. *The orthogonal projection of uniform circular motion on any diameter of the circle is called Simple Harmonic Motion.*

To find the velocity and acceleration of a point moving with simple harmonic motion.

Let P be a point moving in a circle of radius a and centre O, with constant angular velocity ω, Op a vector perpendicular to OP, and numerically equal to $a\omega$; then Op represents the velocity of P. Let M, m be the orthogonal projections of P, p, on the diameter AA'. Then M has a simple harmonic motion in AA'.



Now the velocity of P is equivalent to the velocity of P relative to M and the velocity of M. The velocity of P relative to M is perpendicular to AA'. Hence the velocity of M = resolved part of velocity of P parallel to AA', and a similar statement is true for the accelerations.

Hence the velocity of M is represented by Om.

And $\qquad Om=\omega\,.\,PM$ by similar triangles

$$=\omega\,.\,\sqrt{a^2-x^2},\text{ if } OM=x.$$

Also the acceleration of P is $\omega^2\,.\,PO$; therefore that of M is $\omega^2\,.\,MO$, or $-\omega^2 x$.

Thus in simple harmonic motion the acceleration is directed to a fixed point, and is proportional to the distance of the moving point from this fixed point.

This fixed point is called the centre of acceleration. The extreme displacement of the moving point from the centre of acceleration is called the **amplitude** of the motion or "vibration."

The numerical value of the acceleration at unit distance is usually denoted by μ. The acceleration at distance x is then $-\mu x$; if ω be the angular velocity in the circular motion, $\omega^2=\mu$.

The **periodic time** of a vibration is the interval between two successive passages of the moving point through any given position in the same direction. This is equal to the periodic time in the circle, which is $\frac{2\pi}{\omega}$, or $\frac{2\pi}{\sqrt{\mu}}$.

The periodic time of a Simple Harmonic Vibration is thus independent of the amplitude, a most important result.

The **phase** of the motion at any instant is the *fraction* of the periodic time which has elapsed since the point was at its extreme position in the positive direction.

For instance, when the point is at A the phase is zero, when at O the phase is $\frac{1}{4}$ or $\frac{3}{4}$, when at A' the phase is $\frac{1}{2}$.

The time of transit from A to M

$$=\text{periodic time}\times\frac{\text{angle } AOP}{4 \text{ rt. angles}}=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}.$$

These results may be conveniently expressed thus:—Let the time t be measured from the instant when OP, the moving radius of the circle, has travelled through a positive angle ϵ from the positive sense of the diameter on which the motion is projected; then, a being the amplitude,

the displacement at time t is $+\ a\cos(\omega t+\epsilon)$,

the velocity is $-\omega a\sin(\omega t+\epsilon)$,

and the acceleration is $-\omega^2 a\cos(\omega t+\epsilon)$, ω^2 being as before equal to μ the acceleration at unit distance.

The angle ϵ is called the **epoch.** The phase when $t=o$ is evidently $\frac{\epsilon}{2\pi}$, so that the *epoch* may be defined as the number of radians in ($2\pi\times$ the phase at the instant from which the time is measured).

51. The following examples embody the more important points in the elementary theory of harmonic motion. The student will have no difficulty in verifying them.

(1) The resultant of any number of simple harmonic motions of the same period in the same straight line is a simple harmonic motion in that straight line.

The displacements may be taken as $a_1\cos(\omega t+\epsilon_1)$, $a_2\cos(\omega t+\epsilon_2)$, $a_3\cos(\omega t+\epsilon_3)$, etc. .... The resultant displacement is $a\cos(\omega t+\eta)$, where $a^2=(\Sigma a\cos\epsilon)^2+(\Sigma a\sin\epsilon)^2$ and $\tan\eta=\frac{\Sigma a\sin\epsilon}{\Sigma a\cos\epsilon}$.

(2) Uniform circular motion is compounded of two simple harmonic motions of the same period and amplitude, their phases differing by $\frac{1}{4}$, and their directions being perpendicular.

Taking axes Ox, Oy as the directions of the motions, we may write the displacements as $x=a\cos\omega t$, and $y=a\cos(\omega t-\frac{1}{4}.2\pi)$, or $y=a\cos\left(\omega t-\frac{\pi}{2}\right)$. The path of the point in the resultant motion is the circle $x^2+y^2=a^2$; the accelerations are $-\omega^2x$, $-\omega^2y$, which compound into $-\omega^2r$ along the radius of the circle.

(3) The resultant of two simple harmonic motions of the same period, of different amplitudes and in different directions, their phases differing by $\frac{1}{4}$, is motion in an ellipse, the acceleration being directed towards the centre and proportional to the distance of the moving point from the centre. The velocity at any point is proportional to the diameter parallel to its direction.

This is called *Elliptic Harmonic Motion.*

Its properties may be obtained at once from uniform circular motion by orthogonal projection.

Let P, P' be a point and its projection; the displacement of P projects into that of P', and hence the average velocity (v) of P into the average velocity (v') of P'; then prove that as v approaches a definite vector so does v'; the hodograph of P projects into that of P', and a vector in the original plane which represents the acceleration of P into a vector which represents the acceleration of P'.

(4) The resultant of two simple harmonic motions of the same period in perpendicular lines is in general an elliptic harmonic motion.

The displacements may be written $x=a\cos(\omega t+\epsilon)$, $y=b\cos\omega t$. Eliminating t, the equation of the path is seen to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\frac{xy}{ab}\cos\epsilon=\sin^2\epsilon.$$

The acceleration is the resultant of $-\omega^2x$, $-\omega^2y$ parallel to the axes; these compound into $-\omega^2r$, where r is the displacement of the point from O.

(5) The resultant of any number of simple harmonic motions in any directions parallel to a plane, of the same period but of different amplitude and phase, is an elliptic harmonic motion.

This follows from examples (1), (3), and (4) by resolving all the motions in the same two perpendicular directions.

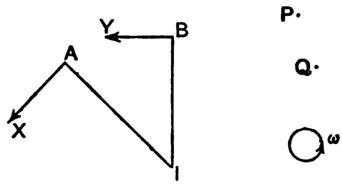
(6) The resultant of two simple harmonic motions of the same phase in different directions, the period of one being twice that of the other, is motion in a portion of a parabola.

The displacements may be written $x=a\cos 2\omega t$, $y=b\cos\omega t$.

The composition of two simple harmonic motions of different periods is beyond the kinematical requirements of this treatise; the curves are of great beauty and of great importance in Physics; in order to trace them the displacements may be conveniently laid off along the axes by means of a table of natural cosines. The reader may consult Thomson and Tait's *Natural Philosophy*, Vol. I. Part i., §§ 67-74, or Clifford, *Dynamic*, Vol. I., pp. 33-37.

52. The theorems given in this and the following section are often of use in kinematical problems.

A number of points $A, B, C, \ldots P, Q$, are moving in a plane in such a manner that the distance between each pair is invariable. To construct the Velocity Diagram of the system.



The angular velocity (ω) of the straight lines joining every pair of points is the same, since all such lines remain at invariable angles to one another.

Since the position vector of Q relative to P is of constant magnitude, the velocity of Q relative to P is $\omega . PQ$ at right angles to PQ. The same is true of every pair of points belonging to the system. Thus the diagram $A, B, C \ldots P, Q$, if rotated through a right angle in the sense of ω is similar and similarly situated to the velocity diagram, the latter diagram having its linear dimensions ω times those of the former.

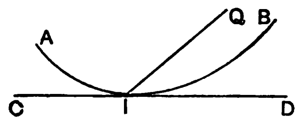
Now let A, B be two points of the system, the directions AX, BY of whose velocities relative to an origin in the plane are known; draw AI, BI perpendicular to these directions, meeting in I. Then from the triangle of velocities AIB the velocity of A must be $\omega . IA$ perpendicular to IA. Therefore from the properties of the velocity diagram the velocity of any other point P of the system is $\omega . IP$ perpendicular to IP.

Thus the motion of the system is the same, for the instant, as

if it were turning round I. I is called the *instantaneous centre* of rotation, which we thus define as a point *on the plane.*

The position of the instantaneous centre can thus always be found if the directions of the velocities of two points of the system are known, and these directions are different. If these directions are the same, and are not perpendicular to the straight line joining the points, I is at infinity and all the points have the same velocity.

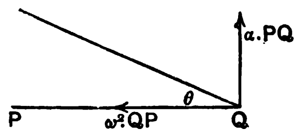
If an area is moving in a plane, one point and one alone is instantaneously at rest, viz. that point which coincides at the instant with the instantaneous centre;* this will often enable us to determine the instantaneous centre.



If a curve AB rolls without slipping on a straight line CD, the instantaneous centre is the point of contact I; for since there is no slipping, the point I *on the curve* has no velocity parallel to CD; also just before the instant considered it was moving *towards*, just after it will be moving *away from* CD; thus its resultant velocity is instantaneously zero.

Hence the velocity of any point Q rigidly attached to the curve is $\omega . IQ$, and QI is a normal to Q's path.

53. A similar theorem holds for the accelerations of the system.

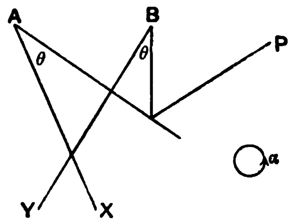


Let ω be the angular velocity, α the angular acceleration of the system. The acceleration of Q relative to P is $\omega^2 . QP$ parallel, $\alpha . PQ$ perpendicular, to PQ; these are equivalent to $\sqrt{\omega^4+\alpha^2} . QP$ inclined at an angle $\theta=\tan^{-1}\frac{\alpha}{\omega^2}$ to QP, this angle being measured in the sense *opposite* to α.

The diagram A, B, C... rotated through an angle θ in this sense is similar and similarly situated to the acceleration diagram.

* If the instantaneous centre is *outside* the area, the point which coincides with it may be supposed rigidly connected with the area.

Let AX, BY be the directions of the accelerations of A, B; draw AJ, BJ making an angle θ with these in the *same* sense as α. J is a point such that the acceleration of any point P of the



system is $\omega^2 . PJ$ parallel and $\alpha . JP$ perpendicular to JP. J is called the *acceleration centre* of the system.

In the particular case in which $\alpha = 0$, J is the intersection of AX, BY.

Examples.

1. A circle is rolling without slipping along a straight line with constant angular velocity. Prove that the velocity of any point of the circumference relative to the centre is numerically equal to the velocity of the centre.

[Both are $a\omega$, where a is the radius of the circle, ω its angular velocity.]

2. Find the radius of curvature of the path of a point on the circumference of the circle.

[Let P be the point, I the point of contact, C the centre of the circle; C's acceleration is zero, therefore C coincides with the acceleration centre. The acceleration of P is $\omega^2 . PC$. The resolved part of this parallel to PI is $\frac{1}{2}\omega^2 PI$, whence the radius of curvature is $2PI$.]

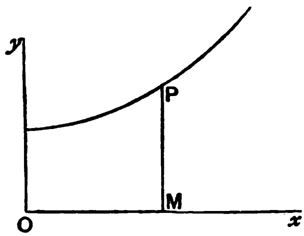
3. A straight rod AB moves with its ends on two fixed intersecting wires OX, OY, the rod turning with constant angular velocity ω. Prove that the acceleration of any point P on the rod is represented by $\omega^2 . PO$, and hence that the motion of P is elliptic harmonic motion.

Prove also that the projections of P on OX, OY, move with simple harmonic motion.

4. If in the above question OX, OY are at right angles, and the angular velocity of AB is not necessarily constant, the speeds of A, B are inversely proportional to their distances from O.

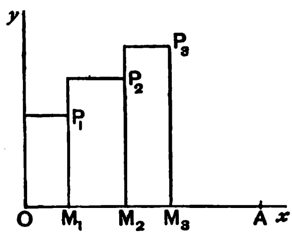
5. The ends of a rod of length l are moving with velocities u, v respectively perpendicular to its length. Find the instantaneous centre of rotation.

54. When the law according to which the speed of a point varies is given, the length of path traversed is often conveniently obtained by means of the **curve of speeds.** Taking



two axes Ox, Oy at right angles, let the abscissa OM be equal to the number of units in the time reckoned from some particular instant, and let the ordinate MP be taken equal to the number of units in the speed at the time represented by OM. As OM increases from zero to any finite length t, P will describe a curve called the *curve of speeds.*

PROPOSITION. *The length of the path described by a point in any time is numerically equal to the area included between the curve of speeds*, Ox, Oy *and the ordinate corresponding to the time.*



Divide the time t, represented by OA, into any number of intervals represented by O_1M_1, MM_2, M_2M_3, ... etc. Let the ordinates M_1P_1, M_2P_2, M_3P_3 ... represent the *average speeds* through each of these intervals; complete the rectangles OP_1, M_1P_2, M_2P_3, etc.

The length of path described in any interval M_2M_3 is accurately equal to the number of units of area in $M_2M_3 \times M_3P_3$, that is, in the rectangle M_2P_3. Therefore the length of path described in time t is accurately represented by the sum of the rectangles. But as the intervals OM_1, M_1M_2, M_2M_3, etc., are indefinitely diminished (their number consequently increasing) the ordinates M_1P_1, M_2P_2 ... have for their limits the speeds at the instants indicated by M_1, M_2, M_3, ..., and the curve through their extremities becomes the curve of speeds. Now by § 7 the limit of the sum of the areas of the rectangles is the area of the curve. Hence the proposition.

In a similar way, if abscissae be taken to represent times and ordinates to represent speed-accelerations, we obtain the **curve of speed-accelerations**, the area of which, estimated as before, is numerically equal to the speed.

Examples.

1. Find by means of the curve of speeds the displacement in time t of a point moving in a straight line with constant acceleration f.

2. If the speed-acceleration $\propto t$, the length of path described from rest $\propto t^3$.

3. If a point moves in a straight line in such a manner that its retardation is proportional to its speed, prove that the space described in any time is proportional to the speed destroyed in that time.

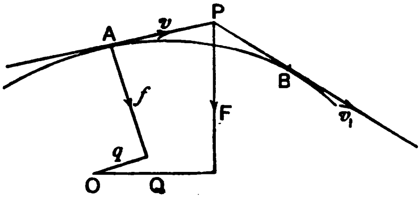
4. What is the curve of speeds for simple harmonic motion?

55. We have defined in § 17 the **moment** of a vector about a point. The moments of the velocity and acceleration of a moving point are important kinematical quantities; in the case of each of these quantities it is to be understood that the vector representing the quantity at any instant is localised at the instantaneous position of the moving point. The moment of the velocity about a point is (§ 47) equal to *twice the areal velocity about that point.*

PROPOSITION. *The rate of change of the moment of the velocity of a point moving in a plane about any fixed point in that plane is equal to the moment of the acceleration about that point.*

Let O be the fixed point, AB the small arc of path described in a small interval τ, AP, BP the tangents at A, B; p, p_1 the

lengths of the perpendiculars from O on these lines, v, v_1 the velocities at A and B, f the acceleration at A, q the length of the perpendicular on the vector $\bar{f}$ localised at A. F the *average acceleration* from A to B.



Then $\bar{v}$ is to be localised at A, $\bar{v}_1$ at B, and $\bar{f}$, as already stated, at A.

The moments of $\bar{v}$ and $\bar{v}_1$ will be unaltered if these vectors are localised at P, instead of at A and B. The sum of the vectors

$$-\bar{v},\quad \bar{v}_1,\quad \overline{F}\,.\,\tau$$

is zero.

Hence (§ 17) if these vectors are all taken as localised at P,

$$vp - v_1p_1 + F\tau Q = 0$$

where Q is the perpendicular from O on the direction of $\overline{F}$ supposed localised at P.

Therefore $$\frac{v_1p_1 - vp}{\tau} = FQ.$$

Now as τ diminishes, $\dfrac{v_1p_1 - vp}{\tau}$ becomes in the limit the rate of change of the moment of the velocity. Further, the point P moves up to A and ultimately coincides with it, and F and Q tend to f and q respectively as their limits. Hence the rate of change of the quantity vp is equal to fq.

This equation in the language of the differential calculus may be written $\dfrac{d}{dt}(vp) =$ moment of the acceleration.

Corollary. If the direction of the acceleration always passes through O, its moment about O is always zero, and vp the moment of the velocity about O is constant. Conversely if the moment of the velocity about O is constant, either the acceleration is zero (*i.e.* the motion takes place with constant velocity), or the direction of the acceleration passes through O.

Examples on Chapter II.

1. If the velocity of a point at time t is the vector sum of $\bar{u}$ and at^{2n+1}, $\bar{u}$ and a being constant vectors and n any integer, the space described between the times $-t$ and $+t$ is $2\bar{u}t$.

2. A straight line AB moves uniformly parallel to itself so as to make constant angles θ, ϕ with two fixed straight lines OA, OB. If C be the third vertex of an equilateral triangle one of whose sides is AB, show that C moves with constant velocity on a straight line through O, and find the ratio of the magnitude of this velocity to the speed of A.

3. A point is displaced from P to Q under constant acceleration; R is the point of the path reached at half-time; V is the middle point of PQ. Prove by a vector method that RV is parallel to the direction of the acceleration, that VR produced intersects the tangents at P and Q to the path in the same point T, and that $TR=RV$.

4. If a point moves in a parabola whose axis is vertical with constant horizontal velocity, show that the vertical component of its velocity increases or diminishes uniformly with the time.

Hence shew that if in rectilinear motion the distance passed over is proportional to the square of the time the velocity will be proportional to the time simply.

5. A point moves from rest in a straight line under the influence of an alternate acceleration and retardation of magnitudes f and f' respectively for equal intervals of time t. Prove that the space passed over in $2n$ intervals is $\frac{nt^2}{2}\{(2n+1)f-(2n-1)f'\}$.

6. AB is a straight line, P and Q are two moving points; P starts from A to travel along AB with speed u and acceleration f and at the same instant Q starts from B to travel along BA with speed u' and acceleration f'; if they pass one another at the middle point of AB and arrive at the other ends of AB with equal speeds, show that $(u+u')(f-f')=8(fu'-f'u)$.

7. Three horses in a field are at a certain moment at the angular points of an equilateral triangle. Their motion relative to a person driving along a road is in direction round the sides of the equilateral triangle (in the same sense) and in magnitude equal to the speed of the carriage. Show that the three horses are moving along concurrent lines.

8. A number of points start from coincidence, with an initial velocity which is the same for all both in magnitude and direction, but with different constant accelerations; if the magnitude of the acceleration is proportional to the cosine of its inclination to a fixed line, prove that at a given instant all the points lie on a

polygon which is always similar to itself, and whose area is proportional to the fourth power of the time. Prove also that this polygon is inscriptible in a circle.

9. Two straight railways converge to a level crossing, at an angle α; and two trains distant respectively a and b from this point are moving towards it with speeds u, v respectively. Find when and where they are nearest each other, and prove that their least distance apart is

$$\frac{(av \sim bu)\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}}.$$

10. Two straight lines of railway cross one another on the same level, and two trains are approaching the crossing, one on each line; if one train has a speed of 30 miles per hour, when the engine is 160 yards from the crossing, and if the greatest speed attainable by it before reaching the crossing is 31 miles per hour and the least is 10 miles per hour, and if the corresponding quantities for the other train are 240 yards, and speeds 40, 42, and 20 miles per hour, prove that a collision is inevitable unless the length of the first train $< 86\frac{1}{10}$ yards or else that of the second $< 90\frac{14}{15}$ yards, the acceleration or retardation of the trains being always uniform.

11. Two rings P and Q connected by a tight inextensible string slide on rods AB, AC inclined at an angle α; if v be the speed of P at any instant and θ the inclination of the string to AC, shew that the speed of P relative to Q is $v\sin\alpha/\cos\theta$.

12. A point P describes a circle of 1 foot radius in 1 hour, and a point Q describes a concentric circle of 4 feet radius in 14 hours; both points move uniformly and in the same sense; shew that the line joining them rotates in that sense for a period of $43\frac{1}{13}$ minutes, followed by a period of $21\frac{7}{13}$ minutes in the opposite sense.

13. It is observed that the direction of the line of smoke from a steamer steaming a miles per hour due north is θ to the east of south; and that the direction of that from a steamer steaming b miles per hour due east is ϕ to the west of south. Shew that if the wind is north-west

$$a\sin\theta(\sin\phi+\cos\phi)=b\cos\phi(\cos\theta-\sin\theta).$$

14. AB is the breadth of a river with parallel banks. A boat whose speed in still water is equal to the speed of the current starts from A so that its bows always point directly towards B. Find the path of the boat and prove that the boat never reaches B, and only reaches the opposite bank after an infinite time.

15. If the hodograph of a motion is a circle described with constant speed, the pole being in the circumference, the path of the moving point is a cycloid.

16. The hodograph of a certain motion is a regular polygon, of which the sides are traversed in succession with the same constant speed. Give a general description of the motion. Prove also that if the pole is at the centre of the polygon the path is closed and the motion periodic.

17. If in the last question the pole is at any other point, and if T be the time taken to describe the polygon, the total displacements in intervals T are all equal and in the same straight line.

18. A point P moves in a circle of radius a in such a manner that the radius vector from P to a point O distant c from the centre of the circle revolves with constant angular velocity ω. Show that the hodograph of the motion may be obtained by subtracting the focal radii vectores of an ellipse of major axis $2\omega a$ and eccentricity $\frac{c}{a}$ from the corresponding radii of a circle of radius $2\omega a$ with the focus for centre.

19. A curve is described with constant speed. Prove that if A, B, C, are three consecutive points such that the time from A to B is equal to the time from B to C, the difference of the vectors AB, BC divided by [the length $AB]^2$ is in the limit a vector parallel to the inner normal and equal to the curvature.

20. Deduce from the properties of the hodograph for motion under constant acceleration that the radius of curvature of the parabola at any point $\propto \operatorname{cosec}^3\theta$, where θ is the inclination of the tangent at that point to the axis.

21. Find the radius of curvature at any point of an ellipse by considering the properties of elliptic harmonic motion.

22. A circle rotates in its own plane about a point in its circumference with angular velocity ω, and a point P moves in the circle so that the radius to P rotates relative to the circle with angular velocity 2ω in the opposite sense. Show that P moves in a straight line and find its velocity.

23. A point P starts from O with constant velocity v in a straight line inclined at an angle α to the straight line OAB through two fixed points A and B. Prove that the angular velocity with which A and B appear to separate, as seen from P, is

$$v \sin \alpha \left(\frac{a}{r^2} - \frac{b}{r'^2}\right),$$

where a and b, r and r' are the distances of A and B from O and P respectively. Show that the points appear relatively at rest after a time $\frac{\sqrt{ab}}{v}$.

24. Two toothed wheels have their axes parallel and revolve in contact; prove that if the common normal to the surface of contact intersects the line of centres in a fixed point, the ratio of the angular velocities is constant.

25. ABC is a right angle; if C begin to move with angular velocity β about B, BC remaining constant, and if B begin to move with angular velocity α about A, AB remaining constant, show that the initial angular velocity of C about A is $\alpha\cos^2 A + \beta\sin^2 A$, all the motion taking place in the plane ABC.

26. AB, BC are two bars freely jointed at B, and pivoted to a fixed point at A. AB rotates round A with one revolution a second; BC rotates round B with three revolutions a second. AB is 1 foot long, BC is 2 feet long, and the whole motion takes place in one plane. Write down the velocity and acceleration of C (giving their directions) when the angle at B is a right angle.

27. Three points are moving with speeds u_1, u_2, u_3 along three straight lines in one plane equally inclined to one another. Show that the speed of their centroid for equal multiples is

$$\tfrac{1}{3}\{u_1^2+u_2^2+u_3^2-u_2u_3-u_3u_1-u_1u_2\}^{\frac{1}{2}}.$$

28. Prove by a vector method that the resultant of any number of simple harmonic motions, in different directions but of the same period and phase, is a simple harmonic motion of that period and phase.

29. Two points are moving each with simple harmonic motion of periodic time t; if they are at rest at the same instants, and the straight line a joining them at one of these instants be at right angles to the straight line b joining them at another, prove that the shortest distance between them is $\dfrac{ab}{\sqrt{a^2+b^2}}$, and that the time required for the distance between them to change from a to the minimum is $\dfrac{t}{2\pi}\sin^{-1}\dfrac{2ab}{a^2+b^2}$.

30. Over an ocean shoal a N. and S. tidal current is superposed on a constant S.E. current whose speed is $\frac{1}{4}$ knot, the maximum speed of the tidal current being $\frac{1}{2}$ knot, its period 12 hours, and its law of variation simple harmonic. Draw the hodograph for a particle of the water, indicating the direction and magnitude of the resultant velocity at intervals of three hours.

31. A caterpillar crawls in a straight line, its head and tail being alternately at rest for equal intervals of time; while the head is at rest the tail moves from rest to rest with a motion which may be regarded as simple harmonic of half a period; and while the tail is at rest the head moves forward with a similar motion; what is the motion of any other point of the caterpillar relative to the head or the tail, assuming that the point always divides the caterpillar's length in the same ratio?

32. Prove that the resultant of two simple harmonic motions in the same straight line of equal amplitude a and of very nearly equal period may be regarded as a simple harmonic motion whose amplitude varies slowly from a to zero, and from zero to a.

33. A common method of compounding simple harmonic vibrations is the following: a platform, to which is affixed a piece of paper, performs simple harmonic vibrations to and fro, while a pen resting on the paper performs simple harmonic vibrations of equal amplitude across the platform, the period of each vibration being adjustable; if the machine be adjusted so that the trace on the paper would be a circle, and then the adjustment is very slightly disturbed so that the periods are no longer exactly equal, sketch the general nature of the trace on the paper.

34. A body is projected vertically upwards with velocity v; after a time t a second body is projected vertically upwards with a velocity v' less than v. If the first body meets the second as soon as possible after the first starts, prove that $t=\dfrac{v-v'+\sqrt{v^2-v'^2}}{g}$.

35. Two men A and B stand at a distance c apart. B starts to move in a direction which he never varies with constant speed v. A starting simultaneously tries to catch B, and moves with a speed which he can vary at will between u and u'. Prove that if A never varies the direction of his motion after starting, and if $v<u'<u$, B will be safe from capture except while crossing a certain space whose area is

$$\pi v^2c^2\frac{(u^2-u'^2)(u^2u'^2-v^4)}{(u^2-v^2)^2(u'^2-v^2)^2}.$$

36. Two boats each move with speed v relative to the water, and both cross a river of breadth a running with uniform velocity V. They start together, one boat crossing by the shortest path, the other in the shortest time. Prove that the difference between the times of arrival is either

$$\frac{a}{v}\left\{\frac{V}{(V^2-v^2)^{\frac{1}{2}}}-1\right\},\text{ or }\frac{a}{v}\left\{\frac{v}{(v^2-V^2)^{\frac{1}{2}}}-1\right\},$$

according as V or v is the greater.

37. Three points move in a plane, starting at the same instant from the same point. The two first have the same given acceleration in the same direction HK; their initial velocities are also given, in given different directions. The initial velocity of the third and its acceleration are wholly perpendicular to HK. Find these latter, so that the three points may be collinear throughout the motion.

38. A point has three independent accelerations, proportional respectively to its distances from the angular points of a triangle ABC, and directed towards these points, all being of the same magnitude at unit distance. It moves in a path which touches the three sides of the triangle. Prove that the points of contact are the middle points of the three sides, and that the speeds at these points are proportional to the sides.

39. A vessel is often steered for a distant light by keeping the forward rigging on one side, the light and the steersman's eye in a straight line. What is the real path of the vessel?

If the distance from the steersman to the mast opposite the rigging be 32 feet, the breadth of beam at the mast 8 feet, and the distance from the light when sighted 2000 yards, prove that at a distance of 200 yards from the light the vessel will have altered her course by (roughly) $1\frac{1}{2}$ points.

40. If, in the last question, the speed of the vessel is constant, prove that the rate at which its course is altered is inversely proportional to its distance from the light.

41. A rod AB is in motion so that the end B moves with constant speed u in a circle centre C, while the end A moves in a straight line passing through C. If $AB=BC=a$, and $AC=x$, show that the speed of $A=\frac{u}{a}\sqrt{4a^2-x^2}$.

42. The piston end of the connecting-rod of an ordinary steam engine is moving with speed v when the rod is at right angles to the crank. Prove that if l be the length of the rod and c that of the crank, the normal acceleration of the crank-pin is

$$\frac{l^2v^2}{c(l^2+c^2)}.$$

43. Three trains of lengths a, b, c feet are travelling with uniform velocities u, v, w feet per second in the same direction on equidistant parallel rails, with their rearmost carriages in a straight line. Show that the trains may all be cut by some straight line or other for a time

$$\frac{a+c}{2v-u-w} \text{ seconds, or } \frac{2b}{u+w-2v} \text{ seconds,}$$

according as $2v \gtrless u+w$.

44. ABC is a triangle; it is moved into any other position $A'B'C'$ in its own plane. E, F, G are the middle points of AA', BB', CC'. Show that EFG is a triangle, the sides of which are perpendicular and proportional to the relative mean velocities of displacement of the points ABC in their motion to $A'B'C'$. Deduce a diagram for the mean velocities of displacement in any finite uniplanar motion, and proceeding to the limit find a construction for the diagram of actual velocities.

45. A man, who can throw a stone up a height of 121 feet, goes down a coal-pit at the rate of $3\frac{3}{4}$ miles an hour. When he is $106\frac{89}{256}$ feet from the top, he throws a stone up the pit-shaft. Show that it will pass him in $2\frac{59}{64}$ seconds after it reaches the top; and find the depth of the pit if he has still $797\frac{217}{256}$ feet to descend when the stone reaches the bottom. [$g=32$ f.s.s.]

CHAPTER III.

THE LAWS OF MOTION.

56. HITHERTO we have dealt with the geometrical aspects of motion only. We now proceed to consider the circumstances under which the motion of *bodies* takes place. The branch of science which treats of these circumstances is called **Kinetics.**

If P, Q be any two points of a disc or lamina moving in one plane, we have learnt in the last chapter (§§ 52, 53) that the magnitude of the vector difference of the velocities or of the accelerations of P and Q is proportional to the length PQ. Hence by sufficiently diminishing the linear dimensions of the disc, the velocity or acceleration of all points on it may be made to approach as nearly as we please to a common velocity or acceleration. Similar considerations apply to the motion in three dimensions of a solid body. This suggests the following definition:

A body so small that, for the purposes of our investigation, the distances between its several parts may be neglected is called a **particle.**

A particle so defined may have an appreciable volume, but its motion can be represented diagrammatically by that of a point.

In certain astronomical investigations even the Sun and planets are regarded as particles. But no body, however small, can be regarded as a particle when its rotation is considered.

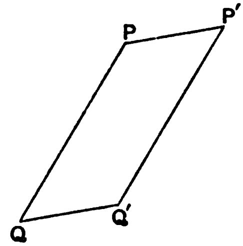
Of a particle as here defined our senses may have direct cognisance; the rules laid down for its behaviour must not be held necessarily to apply without further discussion to the atoms and molecules, of which physicists and chemists suppose bodies to be built up.

A body the distances between whose several parts cannot be neglected may be regarded as made up of particles.

If, during the motion considered, every straight line joining a pair of the particles of the body remains of constant length, the body is called a *rigid* body.

To secure this it is only necessary that each particle of the body should maintain constant distances from any given three which are not in one straight line.

If, in addition, each straight line joining a pair of particles preserves a *constant direction* during the motion, the body is said to have *a motion of translation only*. In this case all the particles of the body describe similar, similarly situated, and equal curves.



For let PQ, $P'Q'$ be successive positions of a line joining any pair of particles. The vectors PQ, $P'Q'$ are identical; so therefore are the vectors PP', QQ'.

Thus, in this case, to describe the motion of the whole body, it is sufficient to describe the motion of a *single particle*.

57. We will now discuss the Laws of Motion,* or Kinetic Axioms, as given by Newton. These axioms were suggested, in part at least, by experiment. Newton in his *Scholium* to the laws gives full credit to Galileo, Christopher Wren, Huyghens, and Wallis in this respect. The really convincing evidence of their accuracy and sufficiency is as follows: On the laws of motion is based the whole science of Dynamics, which, with the Newtonian Law of Gravitation, enables us to predict the relative motion of the bodies of the solar system; these predictions are daily and hourly fulfilled with remarkable exactitude. Thus the laws of motion describe with great accuracy the phenomena with which they deal and satisfy the only valid test for the sufficiency of a scientific doctrine.

58. The First Law of Motion. *Every body continues in a state of rest, or of uniform motion in a straight line, except in so far as it is compelled by forces* † *to change that state.*

* I wish to acknowledge the valuable help I have received, in dealing with the Laws of Motion, from Mr. W. H. Macaulay of King's College, both in conversation and also from his article in the *Bulletin of the American Mathematical Society, July*, 1897; but I must assume full responsibility for the final shape in which the interpretation of the Laws appears. I have also freely consulted Mach's work, *Die Mechanik in ihrer Entwickelung*, Clifford's *Dynamic* (Vol. II.), and Karl Pearson's *Grammar of Science*.

† Newton's equivalent for the word *force* in its modern sense is "vis impressa."

Uniform motion in a straight line we have agreed to call constant velocity. To estimate velocity, and to test its constancy, we must be able to measure *time*. The measurement of time we regard, following Newton, as independent of the First Law of Motion. (See § 4, Chapter I.)

The word *body* in the above statement of the law may for our purposes be regarded as equivalent to *particle*, though Newton gave it a somewhat wider significance. We learn then from the law that *a particle whose velocity is not constant*, i.e. *which has acceleration, is acted on by force.*

Before enunciating the law Newton defines *Force* (*Principia*, Def. iv.) as "an action exercised on a body with a view to changing (*ad mutandum*) its state of rest or uniform motion in a straight line." He then catalogues the various sources of force (*ex ictu, ex pressione, ex vi centripetâ*, the latter having reference chiefly to gravitation in Newton's work). In every case it becomes evident that the circumstance determinative of acceleration in a body is the presence of another body or bodies.

In dealing with the motions of two or more bodies, though, as we shall see, each influences the other, it is often convenient to confine our attention to the motion of one of the bodies only, merely *stating* the effect of the other bodies on it, without attempting to realise in thought their actual motions. To enable us to make this statement compactly the concept of force is employed; we may thus expect to find force presently defined in such a manner as to determine quantitatively the acceleration of a particle regarded as due to other bodies.

When the motion of a body is modified by the hand or any part of the human system, the strain or deformation of the part of the system in contact with the body is accompanied by physiological phenomena which the brain interprets as a sense of *effort*; hence has arisen the idea of force as something external to bodies which modifies their motion; no such assumption is necessary to Kinetics.

59. There remains another important point for discussion. Velocity, as we have seen, is always *relative*; and the question arises: To what origin and axes are the changes of velocity which are to be regarded as indicative of force to be referred? The particular origins and axes used, though not explicitly defined in the law, are yet implied in the history of it.

Galileo was the first to investigate with any success the laws of motion; the motions he studied were those of bodies relative to the Earth's surface at a particular spot; from Galileo's limited point of view, then, a point on the Earth's surface may be taken as origin, and a vertical line (defined by the direction taken by a

plumb line), a north and south line and an east and west line through this point as axes, relative to which the accelerations are to be measured. These we may call for brevity **Galileo's axes.** Certain refined experiments, however, such as that known by the name of Foucault's Pendulum, experiments with gyroscopes, and experiments carried out in a mine at Freiberg in Saxony with falling bodies show that for many purposes Galileo's axes are inadequate, and that the First Law represents observed facts more accurately when the motion is referred to a point on the Earth's surface (or to the Earth's centre) as origin, and to axes determined in direction by lines drawn to the so-called "fixed" stars.

The difference between the motion of a body near the Earth's surface as predicted by the laws applied to Galileo's axes and its *observed* motion relative to these axes is in general very small; *e.g.* if a stone is dropped at the Equator from a height of 100 feet in a vacuum, the observed and predicted places at which it strikes the ground are only about $\frac{1}{5}$ of an inch apart.

A still closer approximation to facts would be obtained by taking for origin a certain point called the Centre of Mass of the Solar System, to be presently defined, the axes as before being determined in direction by lines drawn from this point to the fixed stars. These axes may be called **Newton's axes.** (See below, §§ 66, 85, 90.)

Most of the motions we shall discuss will be motions of bodies at or near the Earth's surface referred to Galileo's axes; in considering these motions, the effects produced by the bodies of the solar system other than the Earth are neglected as too minute for observation, even those produced by the Sun and Moon only becoming important in certain special cases such as that of the tides. The accelerations produced on one another by particles situated at a given spot on the Earth's surface, according to Newton's *Theory of Universal Gravitation*, are also too small to be taken account of except in certain delicate experiments.

60. On Mass. Newton defined mass as the quantity of matter in a body. Apart from metaphysical difficulties, the definition fails to suggest how matter should be measured in the case of bodies of different material; it does not tell us how the amount of matter in a piece of iron, for instance, is to be compared with that in a piece of wood. He further attributes to matter a property called *inertia*, of which he remarks: "through the inertia of matter it comes about that every body is disturbed with difficulty from its state of rest or motion";

the more matter in a body, according to Newton, the more inertia it will have, that is, the more difficult will it be to set it in motion with a given speed. That bodies *do* differ among themselves in the relative ease with which they may be set in motion the roughest observations tell us; for instance, two barrels, both alike to the eye, but one empty and the other full, are easily distinguished when an attempt is made to set first one and then the other in motion; and that this phenomenon has nothing to do (except indirectly) with the *weight* of the bodies may be inferred from the fact that difficulty is experienced in setting in motion a perfectly balanced fly-wheel, the *weight* of which is of course taken by its supports.

To give a measure of this property of *inertia*, then, was the object of Newton's concept of *mass*. The view of mass that we shall adopt renders it necessary to discuss at this point certain phenomena usually considered in connection with the Third Law of Motion.

Confining ourselves for the present to Galileo's axes, experiments show that, when two particles are found to be directly affecting each other's motion, this always happens in such a way that each particle induces an *acceleration* in the other, that these accelerations are in the line joining the particles * and of opposite senses, and that their magnitudes are *in a constant ratio* for the same two particles, whatever be the nature of the link between them; whether, for instance, it be a mechanical connection, such as a tight string, or actual contact, or gravitational (as in the case of Cavendish's experiment), electrical or magnetic. The acceleration so induced in a particle will in general be one component only of its resultant acceleration; for example, in the case of two particles moving in a straight line, and connected by a tight string, the accelerations of the particles must have some common value f; suppose that if the string were cut the particles would have accelerations f_1 and f_2; then it is the ratio $f_1-f:f-f_2$ which is constant for the pair of particles. We make then the following definition:

The mass-ratio of two particles is the negative inverse ratio of the magnitudes of their mutually induced accelerations.

Since these accelerations are always in opposite senses, the ratio of their magnitudes is negative, and all mass-ratios are *positive*.

* The phenomena of Electrodynamics form an exception to this rule as to direction.

If now we choose a certain particle A as being of unit mass, a particle B whose mass is m is such that

$$\frac{\text{magnitude of acceleration of } A \text{ due to } B}{\text{magnitude of acceleration of } B \text{ due to } A} = \frac{m}{1}.$$

Thus the mass-ratio of the particle B to the unit particle A is a number m, which we may call the mass of B.

Similarly, in the case of another particle C, the mass-ratio of C to A is a number m', which we may call the mass of C.

If now the particles B, C be tested by experiment, we find, as previously remarked, that their mass-ratio is a constant number.

Moreover, a comparison of experimental results shows that this number is equal to $\frac{m}{m'}$.

The same is, of course, true of any other particles.

Thus there is associated with each particle a certain number called its *mass*, such that the ratio of the masses of any two particles is equal to their mass-ratio as above defined.

When two or more particles are rigidly connected so as to form a single particle, the mass of the single particle is found to be equal to the arithmetic sum of the masses of the separate particles composing it. For instance, the mass of a leaden bullet A, formed by melting together two other leaden bullets B and C, is the sum of the masses of B and C.

Two homogeneous particles of the same material and volume are found to have equal masses. Hence the masses of homogeneous particles of the same material are proportional to their volumes, a result in accordance with observed facts.

The mass of a particle, as is seen from the above discussion, is a constant scalar quantity. The *constancy* of mass has usually been assumed owing to Newton's definition of it.

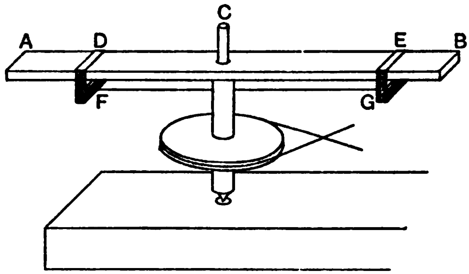
Units of Mass. The British unit of mass, *i.e.* the body regarded as a particle whose mass is arbitrarily assumed to be unity, is a lump of platinum kept at Westminster, and called the Imperial Standard Pound.

The C.G.S. unit of mass is $\frac{1}{1000}$ part of the mass of a lump of platiniridium kept at Paris, and called the International Prototype Kilogram. This unit is called a Gram.

The Kilogram was designed to represent, and approximately does so, the mass of 1000 cubic centimetres of distilled water at 4° C. and 760 millimetres barometric pressure.

Our definition of mass is now complete as far as Galileo's axes are concerned; we shall presently extend it.

61. The mass properties of bodies may be demonstrated roughly for lecture purposes by means of the following apparatus:



A bar AB of polished steel, slightly oiled, and about 1 metre long, 2·5 cm. broad, and as thin as is consistent with rigidity, is attached at its centre to a vertical spindle C so that the bar is at right angles to the axis of rotation of the spindle. To the rod, and sliding on it on opposite sides of C, are attached, by flat, closely fitting stirrups, two carriers D, E of thin metal; they can be loaded with shot, and can be connected by a stout thread FG of twisted silk, which passes freely through a hole in the spindle; the length FG is adjustable. At A and B are two stops, not shown in the figure, to prevent the carriers from flying off the bar. The bar and spindle can be set in rapid rotation by means of a multiplying apparatus, and provided the metal parts are sufficiently bulky no difficulty is experienced in maintaining a sensibly uniform rate of rotation. If the carriers are placed in any positions, and not connected by the thread, they will, when the bar is set rotating even at a low speed, fly with some violence to the ends. If however the carriers are connected by the thread, it will be found that, when the thread is tight and the bar is made to rotate, there is a definite position in which they will remain in equilibrium relative to the bar, and that, as the speed of rotation is gradually increased, this equilibrium will not be disturbed till a very high speed is attained; the disturbance is then of course due to imperfect adjustment, vibration of the apparatus, or other subsidiary causes. Suppose that the bar is rotating with constant angular velocity ω, and that the carriers are in equilibrium when distant respectively x and y from the spindle; then the carriers may be regarded as particles moving with uniform circular motion, and their accelerations towards the spindle are $\omega^2 x$ and

$\omega^2 y$; that these accelerations are almost entirely due to the connection is inferred from the fact that the limits between which relative equilibrium is possible for given loads of the carriers are very narrow indeed and that the position of equilibrium does not depend on the value of ω. The ratio $\frac{x}{y}$ is thus approximately the mass-ratio of the carrier E to the carrier D. Apart from the friction, the fact that the carriers are not particles introduces a slight uncertainty into the measurements.

By altering the lengths x and y, or the angular velocity, keeping the loads of the carriers unaltered, we can establish the existence of a mass-ratio between two bodies regarded as particles.

By experimenting with different loads A, B, C, we can establish the equation

$$\begin{aligned}\text{Mass-ratio of } A \text{ to } B &\times \text{mass-ratio of } B \text{ to } C \\ &= \text{mass-ratio of } A \text{ to } C.\end{aligned}$$

Finally, by using loads large in comparison with the masses of the carriers we can establish the scalarity of mass by finding the mass-ratio of a load A to a load C, that of a load B to the load C, and lastly that of a load $A+B$ to the load C.

62. Mass-Acceleration and Momentum. It is clear from our definition of mass that the product of the acceleration of a particle and the number representing its mass plays an important part in the study of its motion. We make then these following definitions:

The product of the mass and acceleration of a particle is called the **mass-acceleration** *of the particle.*

The product of the mass and velocity of a particle is called the **momentum** *of the particle.*

The mass of any particle being a constant scalar, and acceleration and velocity being vectors, mass-acceleration and momentum are also vectors. Further, a particle may be said to have any number of *independent* mass-accelerations or momenta in the sense that it may have independent accelerations or velocities, each independent mass-acceleration or momentum being the product of the corresponding acceleration or velocity and the mass of the particle. The utility of this conception will be soon seen.

Again, because mass is a *constant* scalar, and the rate of change of velocity of a particle is equal to its acceleration, it follows

that the rate of change of *momentum* of a particle is equal to its mass-acceleration.

The unit of mass-acceleration is the mass-acceleration of a particle of unit mass moving with unit acceleration.

The unit of momentum is the momentum of a particle of unit mass moving with unit velocity.

Examples.

1. A mass of 10 tons is uniformly accelerated in the direction of its velocity, and its speed changes from 20 to 60 miles an hour in 16 seconds. Express in ft.-lb.-second units its initial and final momenta, and its mass-acceleration.

2. A mass of 1 kilogram falls from rest under gravity for 10 seconds. Express in C.G.S. units its momentum at the end of that time, and its mass-acceleration.

63. The Second Law of Motion. *The rate of change of momentum of a particle measures the force acting on it, and the direction of the change of momentum is the direction of the force.*

Or, in accordance with the last section, the force acting on a particle is measured by its mass-acceleration.

This so-called *Law* is thus primarily a definition of force as a quantity. Let a mass m have an acceleration f; then the force acting on it is said to be mf, or if P denote the magnitude of the force,

$$P = mf \text{ units of force.}$$

Putting $m=1$ and $f=1$ in this formula, we see that **the unit force is that force which produces unit acceleration in unit mass.**

It is thus a *constant* independent of the value of the acceleration due to gravity at a given spot.

The specific units in use are:

1. In the British system, the *poundal.*

A mass of 1 lb. falling freely under gravity near the Earth's surface is said to be acted on by its *weight alone.* It will have g units of acceleration, and therefore g units of force are acting on it. g poundals therefore go to the weight of a pound.

In practice the *weight of a pound* is often used as unit; this, however, containing as it does g poundals, varies from point to point of the Earth's surface.

2. In the C.G.S. system the unit is called the *dyne*; reasoning similar to the above will show that the weight of a gram contains approximately 981 dynes. ($g=981$ cm.s.s. approximately.)

As a consequence of our definition of force, all forces must be expressed in the British system in *poundals*, in the C.G.S. system in *dynes*, *otherwise the relation* P = mf *is not necessarily true.**

The poundal and the dyne are called *absolute* units of force.

Examples.

1. A force equal to the weight of 20 lbs. acts on the mass of a ton for 2 minutes. If the mass is initially at rest, find its acceleration and the speed produced.

[Let f be the acceleration. The mass-acceleration is $2240f$. The force is $20g$ poundals. Hence $20g = 2240f$, or $f = \frac{2}{7}$ f.s.s. Speed at end of 2 minutes $= \frac{2}{7} \cdot 120$ f.s. $= 34\frac{2}{7}$ f.s.]

2. Find the number of dynes in a force which will produce in a kilogram a velocity of 20 cms. per second, the force being constant and applied for 25 seconds.

3. A mass of 1 gram vibrates through a millimetre on each side of its mean position 256 times per second. Assuming the motion to be simple harmonic, find the maximum force on the mass in grams' weight, taking $g = 981$ cm.s.s.

4. A cannon ball of mass 10,000 grams is discharged with a velocity of 45,000 cms. per second from a cannon the length of whose barrel is 200 cms. Prove that the mean force exerted on the ball during the explosion is $5{\cdot}0625 \times 10^{10}$ dynes.

64. Principle of the Physical Independence of Forces. *If any number of forces act on a particle, each produces its own acceleration in its own direction independently of the effect of the other forces or of the motion the body already has.*†

That is, if a set of circumstances A when existing alone produce an acceleration $\bar{f}$ in the particle, and another set of circumstances B when existing alone produce an acceleration $\bar{f'}$ in the particle, the sets A and B if existing together will produce an acceleration which is the resultant of the *independent* (§§ 29, 39, Chap. II.) accelerations $\bar{f}$ and $\bar{f'}$ in addition to any other motion the particle already has.

* If the law be enunciated, 'Rate of change of momentum is *proportional to* the force'... etc., its algebraical equivalent is $P = Kmf$, where K is a constant. The use of the above units of force reduces K to unity.

† This principle, together with the statement of the law we have given in § 63, is generally regarded as being implied in Newton's Second Law, which he gives as follows: "Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur." In amplification of this Newton formulates the principle.

The principle is most important as showing that all the motions produced in a body by various forces are *independent* in the sense in which the word is employed in Chap. II., and that, therefore, the displacements and velocities resulting from each motion may be calculated separately, and the results combined by the vector law.

Thus a particle projected in any direction under gravity near the Earth's surface has a vertical acceleration numerically equal to g, independent of the velocity of projection, and will therefore [§ 40 (iii.)] describe a parabola.

The utility of the definition of momentum and mass-acceleration as vectors is now obvious.

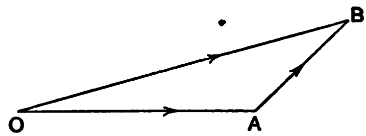
65. The principle forms *Newton's basis for the Parallelogram of Forces*, which states that two forces acting on a particle may be combined according to the same rule as other vectors.

Definition. *The resultant of any number of forces acting on a particle is that force which produces the acceleration which is the resultant of the accelerations which would be due to each of the forces separately.*

The forces which produce the component accelerations are called the *component forces.*

Proposition. *The resultant of two forces acting on a particle is their vector sum.*

Let m be the mass of the particle. Let OA, AB represent the accelerations which would be due to the forces separately.



$\therefore$ OB represents the resultant acceleration.

$\therefore$ by definition $m.OB$ represents the resultant force in magnitude and direction.

But $m.OB$ is the vector sum of $m.OA$ and $m.AB$, which represent the two component forces. The proposition is therefore established.

This proposition, known as the *Parallelogram of Forces*, is true whether OA, AB represent constant or variable accelerations.

Corollary I. The resultant of any number of forces acting on a particle is their vector sum.

It follows, of course, that the algebraic sum of the resolved parts of the forces in any direction is equal to $m \times$ resolved acceleration in that direction.

Corollary II. If a particle is acted on only by a number of forces whose vector sum is *zero*, its acceleration relative to the axes we are using will be zero, *i.e.* the particle will be at rest, or moving with constant velocity; the same will be true if the same set of forces be applied to a particle of any mass whatever; this is the reason why mass does not enter into statical investigations.

66. We are now in a position to discuss what must be regarded as the most important of Newton's extensions of the fundamental kinetic principles laid down by his predecessors.

Our definition of mass (§ 60) asserts that if m, m' be the masses of two particles, f, f', their mutually induced accelerations relative to Galileo's axes (*to which alone, as yet, our theory applies*), $m\bar{f}+m'\bar{f'}=0$. Now, adopting the language of the Second Law, the product $m\bar{f}$ is the force acting on the mass m due to the mass m', while the product $m'\bar{f'}$ is the force acting on the mass m' due to the mass m, and these are in opposite senses and in the straight line joining the particles.

These opposed forces Newton calls Action and Reaction, and hence in the case of two particles referred to Galileo's axes, we may say in the words of Newton's

Third Law of Motion. *Reaction is always equal and opposite to action; that is to say, the actions of two bodies (particles) on each other are always equal and opposite.*

This law Newton assumes to hold throughout the solar system; hence, though at first sight a mere re-statement in less direct terms of the experimental results embodied in our definition of mass, it must in all its generality be taken to imply

(i.) That the bodies of the solar system have *masses* in a sense analogous to that in which terrestrial bodies referred to Galileo's axes have (for what we really measure in the case of the motion of these bodies are certain velocities and accelerations); and that it is possible to find a set of axes relative to which the accelerations determinative of these masses are to be estimated.

The origin of these axes is a certain point called the Centre of Mass of the Solar System, and the directions of the axes are determined by reference to the fixed stars.

(ii.) Taking into account the principle of the physical independence of forces, the law must be considered as asserting that the acceleration of any particle A relative to the supposed axes can be resolved by the vector law into a number of accelerations f_{AB}, f_{AC}, f_{AD}, ... towards other particles B, C, D, ... and the acceleration of a particle B into f_{BA}, f_{BC}, f_{BD}, ... towards A, C, D, ..., such that the ratio $-\frac{f_{AB}}{f_{BA}}$ is a constant ratio called the mass-ratio of the particles B, A, and that when this is done all the accelerations of the solar system relative to the supposed axes are accounted for; and, further, that the scheme of pairs of accelerations is *unique*.

Finally, we remark that when one set of axes (1) is determined to which the laws apply, any set of axes (2) whose origin moves with constant velocity while the axes retain their direction relative to the axes (1) may be used instead. For, in discussing the force on a particle, we are only concerned with *changes* of momentum, and the change of momentum of a particle relative to each set of axes is the same. This is the substance of Newton's Fifth Corollary to the Laws of Motion. For motions of bodies near the Earth's surface of short duration, Galileo's axes may be regarded as thus moving relative to Newton's axes. We postpone certain further considerations until we have discussed the properties of the Centre of Mass.

Axes to which, in dealing with the laws of motion, the changes of momentum of particles are referred have been usually called "fixed" axes. Mr. Macaulay (*Bulletin of the American Mathematical Society*) suggests that the name *kinetic axes* would be preferable, with the word *provisional* prefixed when necessary, to suggest that the axes so chosen are only one set in a series of approximations. We shall adopt this suggestion in the text. A "fixed" point is then a point fixed relative to the set of kinetic axes under consideration, and a particle *at rest* is *at rest* relative to these kinetic axes. No alteration from common usage will, however, be made in the wording of examples taken from examination papers.

67. We here append, for convenience, a short summary of the kinetic principles hitherto laid down.

(1) **Time.** The measurement of time is independent of the laws of motion (§ 4).

(2) **Law I.** The circumstances under which the motions of particles take place determine *accelerations*. These circumstances are the presence of other particles. The motions are supposed to be referred to certain axes, provisionally to **Galileo's axes.**

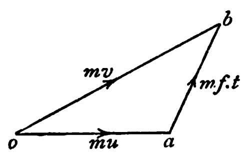
(3) **Mass.** Referring to Galileo's axes, particles have constant *mass-ratios* (inverse acceleration-ratios); taking an arbitrary unit, numbers may be assigned to all particles called their masses.

(4) **Law II.** defines **force** as a quantity. Newton's comments on Law II. amount to the principle of the Physical Independence of Forces; from this principle the Parallelogram of Forces is proved.

(5) **Law III.** implies that, accelerations being estimated relative to **Newton's Axes**, all particles of the solar system have masses, and that if the acceleration of each particle be resolved into accelerations in the lines joining it to every other, the mass-accelerations between any pair are equal, that the system of pairs is unique, and that no accelerations are left unaccounted for.

68. Impulse. As a rule the effect of one particle A on the motion of another B is most conveniently studied by considering the rate of change of momentum produced, *i.e.* the *force* acting on B. In some cases, however, the *whole change of momentum in a definite interval* is all that it is necessary to consider.

DEFINITION. *The change of momentum produced by a force in a given time is called the* **Impulse of the force** *for that time.*



Thus if a force P changes the momentum of a mass m from $m\bar{u}$ (represented by oa) to $m\bar{v}$ (represented by ob) in time t, the impulse of the force for time t is the vector difference of the momenta, represented by ab. When the acceleration, and therefore the force, is constant, the numerical value of this vector difference is mft or Pt.

In many cases, such as that of a billiard ball struck by another ball or a cue, finite change of momentum is generated in a particle in a time too short to measure, so that for purposes of observation it may be supposed generated *instantaneously*. In this case the particle is said to be acted on by an **Instantaneous Impulse** or **Blow,** which is thus regarded as the limit of the Impulse of a very large force for a very short time. The word *Impulse* used without qualification is to be understood as meaning *Instantaneous Impulse.*

An instantaneous impulse is thus measured by the change of momentum generated.

The unit impulse is that impulse which produces unit change of momentum.

By the principle of the Physical Independence of Forces, the resultant of the impulses of the forces acting on a particle for a given time is the vector sum of the impulses. Hence, proceeding to the limit,

The resultant of any number of instantaneous impulses acting on a particle is their vector sum.

When a particle is acted on by an instantaneous impulse,

(1) Its position is not sensibly altered during the action of the impulse; for the impulse produces its effect in a time τ which is to be regarded as insensible; if then $\bar{v}$ be the average velocity of the particle during this time τ, the displacement is $\bar{v}\tau$, which is also insensible, since $\bar{v}$ is finite.

(2) In the same way, if a force P of finite magnitude is acting on the particle, the change of momentum due to it during the action of the impulse is insensible; for this change of momentum is $\bar{P}\tau$, which, as before, is insensible.

The student will note that when an instantaneous impulse acts on a particle we have the exceptional case of § 24. The particle cannot be said to have a velocity *at* the instant at which the impulse acts, but it may be said to have one velocity "up to" and another "on from" the instant.

Impulse between Two Particles. Since the action and reaction between two particles are by the Third Law always equal and opposite, the impulses of these forces for any time are always equal and opposite; and hence, taking the limiting case, when the forces become very large and the time very small,

If the action between two particles is of the nature of an instantaneous impulse, the impulses on the two particles are equal and opposite. Or if M, M' be the masses of the particles V, V', the changes of velocity produced by the impulses,

$$M\overline{V}+M'\overline{V'}=0.$$

The effect of a constant succession of impulses on a particle, when the number of impulses is indefinitely increased and their magnitude, as also the intervals between them, indefinitely diminished, approximates to that cf a force. (See Chap. VI., §§ 150, 151.)

Newton called force and impulse alike "vis," and seems to have regarded *impulse* as the more fundamental conception of the two.

Examples.

1. A ten-pound shot is projected from a gun with an initial speed of 1200 feet a second. What is the impulse on the shot, and what constant force applied for half a second, would produce the same momentum?

2. If the mass of the gun in the last question be half a ton, and the gun may be treated as a particle, with what speed does the gun begin to recoil?

3. Two equal particles are initially at rest, the one being acted on by a constant force P, the other by a series of equal impulses $P\tau$ at equal intervals of time τ. Prove that, if the first impulse acts at a time $\frac{\tau}{2}$ after the force begins to act, the particles will have described equal spaces at the end of any number of the equal intervals.

69. Tension of a String. A string of negligible mass and sensibly constant length is called a *light inextensible string.* It is sufficient for many purposes to regard such a string, when straight and connecting two particles, as a mere geometrical connection imposing on the particles the condition that the acceleration of either particle relative to the other, resolved in the direction of the string, is

(1) zero, if the string is not rotating, or

(2) $l\omega^2$, if the string is of length l and is rotating with angular velocity ω.

This at once follows from the fact that the position vector of either particle relative to the other is of constant length. (§ 49, Cor.)

The acceleration of either particle perpendicular to the string is not altered by the connection. The modification of the motion of either particle is conveniently attributed to a *force* called the Tension of the String.

We shall work out the following example in terms of accelerations, and then give the argument in abbreviated form, introducing a force-symbol to represent the tension.

Example. *A light inextensible string connects two masses* m *and* m′ *acted on by known forces* P *and* Q *respectively, in opposite senses, in the direction of the string. To find the tension of the string.*

Let us take the positive direction as from left to right. Then, by Newton's Second Law, if there were *no* string connecting the

masses, the acceleration of m would be $+\frac{P}{m}$ (left to right), and that of m' would be $-\frac{Q}{m'}$ (right to left).

Now, because of the string, both masses have some common acceleration f, which we may suppose to be from left to right.

Writing $$f=\frac{P}{m}-\left(\frac{P}{m}-f\right),$$

and $$f=-\frac{Q}{m'}+\left(\frac{Q}{m'}+f\right),$$

we see that

$-\left(\frac{P}{m}-f\right)$ is the part of m's acceleration due to m',

$+\left(\frac{Q}{m'}+f\right)$ is the part of m''s acceleration due to m.

Then, by Newton's Third Law,

$m\times$ the part of m's acceleration due to m',

$+m'\times$ the part of m''s acceleration due to $m=0$.

$$-m\left(\frac{P}{m}-f\right)+m'\left(\frac{Q}{m'}+f\right)=0.$$

This at once gives $(m+m')f=P-Q$,

or $$f=\frac{P-Q}{m+m'},$$

showing that f is from left to right if $P>Q$, otherwise from right to left.

Now $m\times$ the part of m's acceleration due to m'

and $m'\times$ the part of m''s acceleration due to m

are the equal quantities called the *action* and *reaction* in Newton's Third Law. In this case they are attributed to the *tension of the string*, which we may call

$-T$ (right to left) for the mass m,

$+T$ (left to right) for the mass m'.

We thus have $$-T=-m\left(\frac{P}{m}-f\right),$$

$$+T=m'\left(\frac{Q}{m'}+f\right),$$

both leading to $$T=\frac{m'P+mQ}{m+m'}.$$

We will now work this example in the *abbreviated form.*

Let T be the tension of the string, f the common acceleration of the two masses, reckoned positive if from left to right. The resultant force on m is $P-T$ (left to right).

$\therefore$ by Newton's Second Law,

$$P-T=mf.$$

Similarly, the resultant force on m' is $T-Q$ (left to right).

$\therefore$ by Newton's Second Law,

$$T-Q=m'f.$$

Add these equations,

$$\therefore\ P-Q=(m+m')f, \text{ or } f=\frac{P-Q}{m+m'},$$

$$\therefore\ T=P-mf=\frac{m'P+mQ}{m+m'}.$$

Having now shown the value of the force-symbol as an abbreviation, we shall always adopt it in future.

We shall assume that the tension of a light inextensible string connecting two particles, and passing round a smooth surface, is the same on each particle; this amounts to assuming that the mutually induced accelerations of the particles are still in accordance with Newton's Third Law as far as magnitude is concerned, the acceleration so induced in each particle being of course in the direction of the part of the string attached to it.

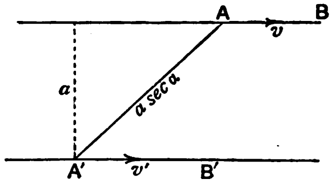
70. When it is necessary to speak of the **tension at any point** of a string whose mass is not negligible, the string may be conceived as a line of particles such that the sum of the masses of the particles in unit length of the string is finite, the number of particles being *ultimately* regarded as increased indefinitely, and the mass of each in consequence diminished indefinitely. When the particles are so linked together that each can only accelerate its neighbour in the line joining them (which is ultimately a tangent to the string), the string is said to be *perfectly flexible*; the links may be conceived to be short light strings as discussed above. The tension of the string *between* two consecutive particles A and B has now a meaning, viz. it is measured by the mass-acceleration either produces on the other. The tension of the string *at* a point P is the limit to which the tension between A and B, taken one on each side of P, approaches when the number of particles is indefinitely increased, and the distances between them indefinitely diminished.

71. Impulsive Tension. When one of two particles connected by a light inextensible string is acted on by an impulse, the string if tight secures that the velocity of each particle relative to the other resolved in the direction of the string is zero both before and after impact. The consequent modification of the motion of each is attributed to an *Impulsive Tension* or *Jerk* of the string, measured of course by the change of momentum each particle owes to the other. Replacing acceleration by change of velocity, the reader will have no difficulty in discussing on the lines already indicated for tensions the jerk at any point of a string whose mass is not negligible.

Examples 1 and 2 which follow will supply an instance of an impulsive tension and a tension.

Example 1. *Two particles of masses* m *and* m′ *are moving in parallel straight lines at a distance* a *apart with given speeds* v, v′ (v > v′) *and are connected by a light inextensible string of given length* a sec α ; *find the motion of each particle immediately after the string becomes tight: find also the impulsive tension of the string.*

Let A, A' denote the positions of the particles as the string becomes tight, AB, $A'B'$ the directions of their velocities just



before the string becomes tight, I the impulsive tension of the string in the sense AA' on the particle A and in the sense $A'A$ on the particle A'.

Since there is no impulse perpendicular to AA', the resolved parts of the velocities in this perpendicular direction will remain unaltered, and therefore after the jerk will be $v\cos\alpha$, $v'\cos\alpha$ respectively; whence the student may prove that the angular velocity of the string after the jerk is $\frac{v-v'}{a}\cos^2\alpha$.

Let u be the magnitude of the common velocity of the particles in the direction of the string after the jerk.

∴ resolving in the sense $A'A$,

$$mu - mv\sin\alpha = -I,$$
$$m'u - m'v'\sin\alpha = I,$$

whence $$u=\frac{mv+m'v'}{m+m'}\sin\alpha,$$

and the velocities are completely determined.

Also $I=\frac{mm'}{m+m'}(v-v')\sin\alpha$ units of impulse.

Example 2. *Two masses of* m *and* m' *lbs. respectively are connected by a light inextensible string which passes over a smooth peg. To find the acceleration and the tension of the string.*

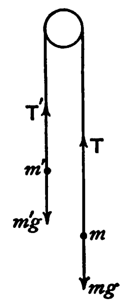
Let $m>m'$. Let T be the tension of the string in poundals, which, as already remarked, we assume to be the same on each side of the smooth peg, f the acceleration in f.s.s., which is evidently the same for both masses since the string continues tight.

The acceleration of m is downwards.

The resultant force on m downwards $=mg-T$ poundals.

$\therefore$ By the Second Law,

$$mg-T=mf. \quad\text{.....................(1)}$$



The acceleration of m' is upwards.

The resultant force on m' upwards

$$=T-m'g \text{ poundals.}$$

$\therefore$ by the Second Law,

$$T-m'g=m'f. \quad\text{.....................(2)}$$

Adding equations (1) and (2), we get

$$(m-m')g=(m+m')f,$$

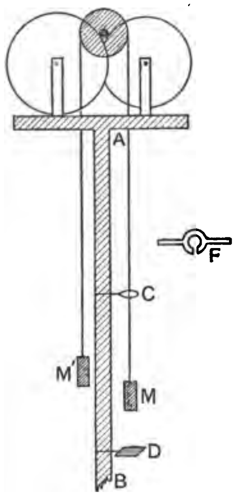
$$\therefore\ f=\frac{m-m'}{m+m'}g \text{ f.s.s.}$$

Substitute this value of f in (1), and we obtain

$$T=\frac{2mm'}{m+m'}g \text{ poundals.}$$

72. The foregoing example illustrates the principle of *Atwood's machine*, by means of which, assuming the truth of the Laws of Motion, a rough value of g can be determined. It consists essentially of a light string passing over a pulley of very small mass and having attached to its ends two equal masses M, M'; the axle of the pulley rests on four wheels called friction wheels, the object of which is to diminish the friction. The motion of one of the masses M is observed by means of a vertical graduated pillar AB, to which are attached by means

of clamps a horizontal ring C and a horizontal platform D, both of which can be fixed at any height.



Several riders of different masses and in the form of a bar (F) too wide to go through the ring C are also provided. A loud ticking *chronometer* is used for measuring the time of motion.

A rider of mass m is placed on the mass M, the system is started from rest opposite the zero of the scale at A, exactly at a tick of the chronometer, and the ring and platform so adjusted by trial that the click of the rider as it is lifted off by the ring and the click of the mass M as it strikes the platform occur simultaneously with ticks of the chronometer; the ear is a very good judge of when this happens.

After the rider is lifted off, the masses M, M' run on with approximately constant speed; the measure of this is obtained at once by measuring CD and dividing the length by the number of seconds taken to traverse it; this speed again divided by the number of seconds taken to traverse AC gives the acceleration of the system during the first part of the motion; g can then be calculated from the formula

$$f=\frac{m}{2M+m}\cdot g.$$

deduced from that of the last article.

The determination of the value of g, however, by Atwood's machine is very far from accurate; the sources of error are the resistance of the air, the mass of the string, friction, and, above all, the mass of the pulley; the difficulty in measuring the short intervals of time can be to some extent overcome; for instance, in the apparatus in use at the Cavendish Laboratory at Cambridge the masses have a possible run of about 30 feet, so that the time of observation is much extended, and an error is of less consequence.

Assuming, however, that the value of g is determined, as is done in practice, by observations on a pendulum, a method which does not involve the *mass* of the swinging body, we may obtain by observing the value of f valuable verification

of the mass property provided we assume that the acceleration which each mass contributes to the other through the string is not affected by the passing of the string round the pulley.

Examples.

1. Devise an experiment to show that mass-ratios exist between three bodies A, B, C.

2. Show in the same way that mass is a *scalar*, *i.e.* that the arithmetic sum of the masses of two particles A and B is equal to the mass of a particle formed by rigidly connecting them.

3. Devise an experiment to prove that weights are proportional to masses.

4. Devise an experiment to show that the tension of a string passing over a smooth pulley is unaltered.

73. Extensible Strings. For our purpose it is sufficient to consider *light* (§ 69) extensible strings.

No string is in nature inextensible; many are by no means so. The following law (called Hooke's Law) is found to hold for *all* light strings, when not stretched beyond certain limits called in each case the "limits of elasticity."

Let T be the tension when the string has a length x, then

$$\frac{T}{\lambda}=\frac{x-x_0}{x_0},$$

where x_0 is the length of the string when its tension is zero, and is called the *natural* or *unstretched length* of the string, and λ is a constant depending on the material of which the string is made. λ is called the modulus of elasticity of the string.

Putting $x=2x_0$, we have $T=\lambda$; therefore λ is the tension of the string when stretched to twice its natural length, *supposing the law to hold for so great an extension.* This it usually does not, the limits of elasticity being passed long before so great an extension is reached. When the law holds for large extensions, the string is called an **elastic string**. Such a string cannot support or give rise to an impulsive tension.

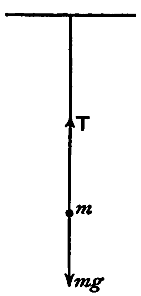
A similar formula holds (within the limits of elasticity) for the tension of a rod or a spiral spring; in the latter cases, however, x may be *less* than x_0, and the tension may be *negative*—that is, the spring or rod may exert a *thrust* whose magnitude is still given by the above formula.

Example 1. *A particle of mass* m *is suspended from a fixed point by an elastic string. Prove that, if the particle is displaced from its position of equilibrium, it will perform simple harmonic oscillations about that position, provided that during the motion the string does not become slack.*

If x be the length of the string when the tension is T, we have

$$\frac{T}{\lambda}=\frac{x-x_0}{x_0}.$$

Let f be the acceleration of the particle *downwards* when the string is of length x.



Then by the Second Law,

$$mg-T=mf.$$

Whence, substituting for T,

$$f=g-\frac{\lambda}{m}\frac{x-x_0}{x_0}. \quad \text{.............(i.)}$$

Putting $f=0$, we have

$$x=x_0\left(\frac{mg}{\lambda}+1\right),$$

which defines the position of equilibrium, in which the tension is just equal to the weight.

Putting a for this value of x, and substituting in (i.), we have

$$f=g+\frac{\lambda}{m}-\frac{\lambda}{m}\cdot\frac{x}{a}\left(\frac{mg}{\lambda}+1\right)$$
$$=\left(g+\frac{\lambda}{m}\right)\frac{a-x}{a},$$

so that the acceleration is directly proportional to the displacement from the position of equilibrium, and, being positive or negative according as x is less or greater than a, is directed towards it. Hence the particle performs simple harmonic vibrations of period

$$2\pi\left\{\frac{ma}{mg+\lambda}\right\}^{\frac{1}{2}} \text{ seconds.}$$

2. With what speed must the particle be projected downwards from the position of equilibrium in order that the string may just not become slack?

3. A particle of mass m is attached to two fixed points A and B on a smooth horizontal table by two equal elastic strings, each of natural length $l\,(2l<AB)$. Find the times of small longitudinal and transversal oscillations of the particle, and prove that the latter is always greater than the former.

74. Reactions of Surfaces. When a particle is moving in contact with the surface of a body, the vector difference of its mass-acceleration from what it would be if the surface were removed, all the other circumstances remaining the same, is attributed to a force called the Reaction of the surface. When a surface can only give a reaction which is entirely along that normal to the surface which passes through the particle, the surface is said to be smooth; no perfectly smooth surfaces exist in nature, but many surfaces are approximately smooth. If the surface can give a reaction which is not entirely along the normal, the surface is said to be rough; the reaction may then be resolved into two components, one along the normal called the pressure and one in the tangent plane to the surface through the particle. This latter component is called the friction, and obeys the following experimental laws:

(1) Its direction is opposite to that of the velocity of the particle relative to the surface.

(2) Its magnitude is a multiple μR of the magnitude R of the pressure; this multiple is approximately a constant, being independent of the relative velocity, and depending only on the nature of the surfaces in contact.

(3) If the particle is at rest relative to the surface, the friction will have the magnitude and direction of the force necessary to keep the particle at rest supposing the surface became smooth. This magnitude must be less than a multiple $\mu_1 R$ of R, where μ_1 is a definite constant a little greater than μ for most surfaces; otherwise relative motion will ensue. When relative motion begins, the ratio of the friction to the pressure decreases very rapidly from μ_1 to μ.

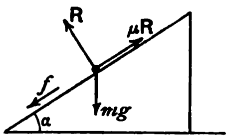
The ratio of the friction to the pressure is called the *coefficient of friction.* If λ be the angle the *reaction* (resultant of friction and pressure) makes with the normal, λ is called the angle of friction. Evidently $\mu = \tan \lambda$.

When the surface is not fixed relative to the kinetic axes, the particle may on its side be regarded as inducing in a small portion of the surface contiguous to it a component of mass-acceleration equal and opposite to the reaction in accordance with the Third Law. But when the surface is *fixed*, we cannot interpret the *force exerted by the particle on the surface* in this way. For instance, if the particle is resting on a smooth horizontal table, the acceleration of the particle perpendicular to the table is zero; and mg being the weight of the particle, R the reaction of the table, we have from the Second Law $R - mg = m \cdot 0$, or $R = mg$, *i.e.* the reaction of the table is equal and opposite to the weight of the particle; but an interpreta-

tion of this result as a particular case of the Third Law in terms of the mass-acceleration of the particle, on the one hand, and that of the system composed of the earth and the table, on the other, cannot be made unless we refer to *Newton's* axes.

75. *A heavy particle slides from rest down a rough fixed inclined plane whose inclination is* a *and coefficient of friction when the particle is in motion* μ. *Determine the motion.*

Let m be the mass of the particle in pounds. It weight is therefore mg poundals.



Let R be the pressure of the plane on the particle, then μR is the force of friction as defined in the last article.

The resultant force *down the plane* is $mg \sin a - \mu R$.

Hence, f being the acceleration,

$$mg \sin a - \mu R = mf.$$

The resultant force perpendicular to the plane is $R - mg \cos a$. But there is no displacement, and therefore no acceleration, in this direction.

Hence $$R - mg \cos a = 0,$$

whence $$f = g(\sin a - \mu \cos a),$$

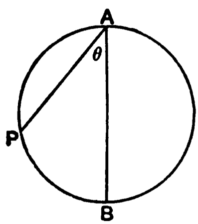
and the displacement from rest in time t

$$= \tfrac{1}{2}gt^2(\sin a - \mu \cos a).$$

Corollary. If $\mu = 0$, the acceleration down the plane $= g \sin a$, and the displacement in time $t = \frac{1}{2}g \sin a \, . \, t^2$.

76. *A heavy particle falls from rest at the highest point of a fixed vertical circle down a smooth chord of the circle. To prove that the time of descent is the same for all chords.*

Let AB be the vertical diameter, AP any chord through A, angle $PAB = \theta$.



Then if t be the time of descent from A to P, as in the last article,

$$AP = \tfrac{1}{2}g \sin(90° - \theta) \, . \, t^2$$
$$= \tfrac{1}{2}g \cos \theta \, . \, t^2,$$

whence $$t = \sqrt{\frac{2AP}{g \cos \theta}} = \sqrt{\frac{2AB}{g}},$$

since $$AP = AB \cos \theta.$$

But this is the time in which the particle would fall freely down the diameter AB, and is independent of the inclination of the particular chord AP.

Corollary. Similarly it may be shown that the time of descent from the circle down all chords to the lowest point is constant.

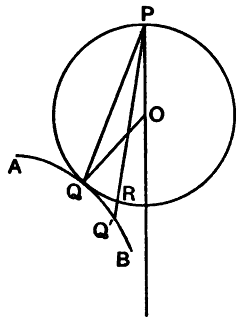
This theorem is due to Galileo.

The following we will leave as an exercise to the student:

The time of descent down all rough chords of a vertical circle, which are drawn through a point at an angular distance λ from the highest point and lie on the side of the vertical through the point opposite to that on which the centre lies, is constant, the coefficient of friction being $\tan\lambda$.

77. Straight Lines of Quickest and Slowest Descent.

(1) *Given a curve* AB *and a point* P *both in a vertical plane; to find the straight line* PQ *drawn from* P *to a point* Q *in* AB *such that the time taken by a heavy particle to slide down* PQ *is a minimum or maximum.*



Draw a vertical circle with highest point at P to touch AB at a point Q. Then PQ is the straight line required.

If PO be the vertical through P, QO the common normal at Q, O is the centre of this circle.

(*a*) If the centre of curvature of the curve AB at Q lies *outside* the verticals through P and Q, the curve in the neighbourhood of Q lies entirely *outside* the circle, and a straight line PQ' drawn to a neighbouring point Q' cuts the circle in a point R between P and Q'.

Thus time down $PQ =$ time down $PR <$ time down PQ', or the time down PQ is a minimum.

(*b*) If the centre of curvature lies *inside* the verticals through P and Q, the curve AB in the neighbourhood of Q lies entirely *inside* the circle, and the time down PQ is a maximum.

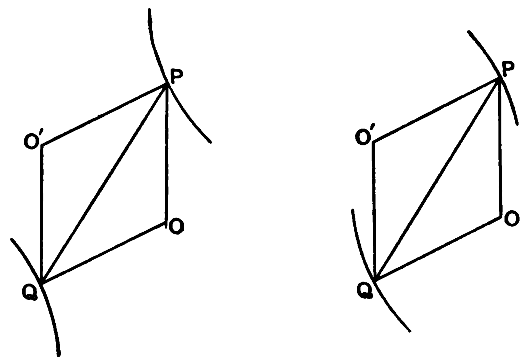
Note that in each case the chord PQ *is equally inclined to the normal at* Q *and the vertical.*

Similar constructions may be made for the chord of quickest or slowest descent from a curve to a point.

(2) *If* PQ *be the straight line of quickest or slowest descent from one curve to another in the same vertical plane, the normals at*

P, Q *to the two curves and the verticals through* P, Q *form a rhombus.*

Let PO, QO' be the verticals, PO', QO the normals. Then since PQ is the chord of quickest (or slowest) descent from P to the lower curve, angles $O'QP$ and $O'PQ$ are equal; and since PQ is the chord of quickest or slowest descent from the upper



curve to Q angles OPQ, OQP are equal. Hence $PO'=O'Q$ and $PO=OQ$. Also angle $O'QP$=angle OPQ by parallels. Hence angle $O'PQ$=angle OQP, and $PO'QO$ is a parallelogram, and therefore a rhombus since $PO=OQ$.

As before, the time down PQ *is a minimum if the centres of curvature at* P *and* Q *both lie outside the verticals through* P *and* Q, *and a maximum if both lie inside. If neither of these conditions obtain, the time down* PQ *thus constructed is neither a minimum nor maximum.*

In many cases the methods of the Differential Calculus are to be preferred to the above construction.

Examples.

1. If two vertical circles touch at their highest or lowest points, the time from rest down the part of any chord through the point of contact intercepted between the circles is constant.

2. Construct the chord of quickest descent to a vertical circle from a straight line in the plane of the circle.

Through Q, the lowest point of the circle, draw QX horizontally to meet the line in X. Mark off XY equal to XQ upwards along

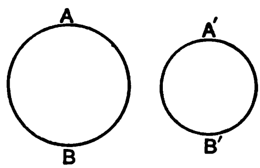
the line. Join YQ, cutting the circle in R. YR is the required straight line. Use the result of Example 1 in the proof.

A similar construction gives the chord of quickest descent from a circle to a straight line.

3. Find the straight line of quickest descent to a given straight line from a given point in the same vertical plane.

This is a particular case of Example 2.

4. Given two non-intersecting circles in the same vertical plane, find the straight lines of quickest and slowest descent from one to the other.



Let A, A' be the highest, B, B' the lowest points of the two circles. Join AA', AB', BA', BB'. Let P, Q be the points in which one of these chords cuts the circles again. Then the normals and verticals at P, Q form a rhombus, as may at once be seen by noting that PQ passes through a centre of similitude.

By applying the criterion of the position of the centres of curvature, we discriminate as follows:

(i.) If the circles are external to each other, the chords through the centre of inverse similitude alone give solutions, and

(a) If B' is above A, the chord $A'B$ gives a minimum, the chord $B'A$ a maximum, time of descent from the circle $A'B'$ to the circle AB.

(b) If B' is below A, but A' above B, the same chords give two minimum solutions, one from the circle AB to the circle $A'B'$, one from the circle $A'B'$ to the circle AB.

(ii.) If one circle is *inside* the other, the chords through the centres of *direct* similitude alone give solutions, both minima, one from the outer circle to the inner, one from the inner circle to the outer.

5. Find the chords of quickest and slowest descent to a given vertical circle from a given point in its plane.

This is a particular case of Example 4.

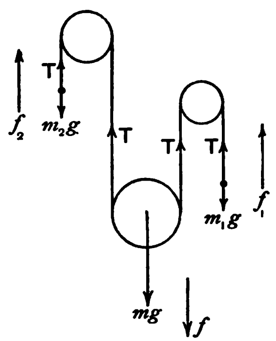
6. If P be a point on a parabola whose axis is vertical and vertex downwards, N the foot of its ordinate, NQ the line of quickest descent from N to the curve, and QM the ordinate of Q, prove that $MN=PN$, provided PN is greater than the latus rectum of the parabola.

7. ABC is a vertical circle, having its highest point at A, its lowest at C. Particles start to slide simultaneously down the chords AB, BC starting from A, B respectively; prove that their shortest distance apart is equal to the perpendicular distance of B from the diameter AC (Wolstenholme).

The diagram of accelerations gives a simple solution.

78. We add several examples illustrative of the motion of particles.

(1) *A string passes over a fixed, under a movable, and over another fixed pulley. To the ends of the string are attached masses* m_1, m_2, *and to the movable pulley a mass* m. *Find the acceleration of the mass* m, *the portions of the string being all parallel and the pulleys without mass.*



Let T be the tension of the string, f_1, f_2, f, the accelerations, v_1, v_2, v, the speeds of the masses m_1, m_2, m, respectively. Without loss of generality we may suppose the senses of f_1, f_2, v_1, v_2, *upwards*; then those of f, v, are downwards.

The actual values of v_1, v_2, v at time t will depend on the speeds with which the masses are started, but since the part of the string between the pulleys is always being lengthened as much as the ends are shortened, we must have $2v = v_1 + v_2$, and since this holds at *each instant of the motion*, we have also

$$2f = f_1 + f_2.$$

The equation given by the Second Law for the mass m is

$$mg - 2T = mf,$$

those for the masses m_1, m_2 are

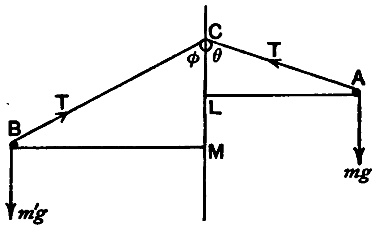
$$T - m_1 g = m_1 f_1,$$

$$T - m_2 g = m_2 f_2.$$

Eliminating f_1, f_2, T, from these we have

$$f = \frac{m(m_1 + m_2) - 4m_1 m_2}{m(m_1 + m_2) + 4m_1 m_2} \cdot g.$$

(2) *Two particles are connected by a fine string which passes over a small smooth pulley in a vertical plane, the pulley being made to rotate with constant angular velocity* ω *about its vertical diameter, carrying with it the vertical plane containing the string and the particles. Determine the configuration of the system when the motion is steady.*



Let m, m' be the masses of the particles, C the pulley, CA, CB the portions of the string, CLM a vertical through the centre of the pulley, AL $(=r)$, BM $(=r')$ perpendiculars on this vertical, T the tension of the string, angle $ACL=\theta$, angle $BCM=\phi$.

When the motion is steady, A and B are describing horizontal circles of radii r, r' respectively. Hence their accelerations are $\omega^2 r$, $\omega^2 r'$ in the directions AL, BM respectively. Resolving horizontally and vertically for each particle

$$T\sin\theta = m\omega^2 r, \qquad T\sin\phi = m'\omega^2 r';$$

$$T\cos\theta - mg = 0, \qquad T\cos\phi - m'g = 0;$$

whence
$$r\cot\theta = \frac{g}{\omega^2} = r'\cot\phi,$$

or $CL=CM$, that is, the particles are in the same horizontal plane,

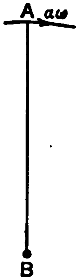
and
$$mr\operatorname{cosec}\theta = m'r'\operatorname{cosec}\phi,$$

or the string is divided by the pulley in the inverse ratio of the masses.

(3) *A particle (mass* m) *hangs by an inextensible string of length* l. *Show that if the upper extremity of the string be suddenly made to describe a circle of radius* a *in a vertical plane with*

angular velocity ω, *starting at the highest point of the circle, the string will initially not become slack unless* $a(l-a)\omega^2$ *exceeds* lg.

Some care is required in writing down the vertical acceleration of the particle relative to the kinetic axes.



Let AB denote the string, A being the upper end.

Since A begins to move at *right angles* to AB there is no *impulsive* tension of the string, and B is acted on merely by the finite forces, T the tension, and mg the weight of the particle. Hence B is *initially* at rest.

The initial velocity of A is $a\omega$ at right angles to AB, and therefore the initial angular velocity of the string is $\frac{a\omega}{l}$.

The acceleration of B relative to A is therefore $\left(\frac{a\omega}{l}\right)^2 . l$, or $\frac{a^2\omega^2}{l}$ in the direction BA (§ 49).

The acceleration of A relative to the kinetic axes is $a\omega^2$ in the direction AB.

Hence the acceleration of B relative to the kinetic axes is $a\omega^2 - \frac{a^2\omega^2}{l}$, or $\omega^2 \frac{a(l-a)}{l}$, vertically downwards.

Hence, equating the mass acceleration of m to the resultant force vertically downwards,

$$m . \frac{a(l-a)}{l}\omega^2 = mg - T.$$

The condition that the string should not become slack is that the tension should be *positive*, giving

$$\frac{a(l-a)}{l}\omega^2 \not> g,$$

or,

$$a(l-a)\omega^2 \not> lg.$$

(4) Is it possible to move A in such a manner as to give B an initial acceleration perpendicular to AB?

79. The following are easy miscellaneous examples on the motion of particles; more difficult examples will be found at the end of the chapter.

Examples.

1. The coupling chain between an engine and a train the mass of which is 96 tons can bear a tension equal to the weight of 12 tons. Treating the train as a particle, find the shortest time in which a speed of 30 miles per hour may be safely attained on a smooth level line.

2. A small bullet is fastened to the end A of a stiff elastic rod AB without mass, and it is observed that when the end B of the rod is held horizontal the bullet weighs down the end A an inch and a half. The whole is then placed on a smooth table, and the end B is held tight. Assuming that the horizontal oscillations of the bullet, when drawn aside and released, are simple harmonic, prove that $8/\pi$ oscillations will be performed in a second.

3. Assuming the Earth to be a sphere of 4000 miles' radius, and that a ball starts from rest at some point on a material tangent plane to the Earth's surface not far from its point of contact with the Earth, shew that it will move along the plane with a simple harmonic motion of period nearly equal to 1 hour 25 minutes.

[The motion is to be referred to "Galileo's Axes," and the acceleration due to gravity is to be assumed to be normal to the sphere at every point, and equal to 32 f.s.s.]

4. A gun weighing 1 ton fires a projectile weighing 7 lbs., and the recoil carries the gun up a smooth inclined plane to the height of 4 feet. Treating the gun and its carriage as a particle, find the initial speed of the projectile.

5. An india-rubber ball is dropped vertically from the hand and bounces a number of times on the ground. Prove that the time-average of the impulses of the ground on the ball is numerically equal to the weight of the ball.

How will the resistance of the air, if taken into account, affect this result?

6. Prove that in the case of a heavy particle sliding up or down a rough inclined plane, the speed at any point will be that due to falling freely under gravity to that point from a certain straight line which slopes downwards in the direction of motion at the angle of friction to the horizontal.

7. The cage of a coal-pit is lowered for the first third of the shaft with constant acceleration; for the next third it descends with constant velocity, and then a constant retarding force just brings it to rest as it reaches the bottom of the shaft. If the time of descent is equal to that taken by a particle in falling four times the whole depth, prove that the pressure of a man inside on the bottom of the cage was at the beginning $\frac{23}{48}$ of his weight.

8. A smooth wedge with one face vertical and the opposite angle $30°$ is fixed on a table with the vertical face projecting over the edge. Three particles of equal mass m are knotted together at equal intervals (a) on an inelastic string, and placed close together at the foot of the wedge. The string is carried over the top of the wedge, and attached to a mass m which hangs freely. Prove that the system after starting into motion will, if the wedge is of sufficient height, come to rest again after a time $11\sqrt{\dfrac{a}{2g}}$; but that

if the height of the wedge is less than $\frac{5}{4}a$, the single particle will pull all the others over the top.

9. A given point lies in the same vertical plane with a circle of radius a at a greater height than any point of the circle; t_1 and t_2 are the least and greatest times of descent down straight lines from the given point to any point of the circle; prove that

$$\frac{1}{t_1^2}-\frac{1}{t_2^2}=\frac{ga}{c^2-a^2},$$

c being the distance from the point to the centre of the circle.

10. If the weights in an Atwood's machine are Mg, $(M-p)g$, and the bar-weight $2pg$ is alternately caught up and deposited on the ring by the smaller weight, show that the intervals between the catching up and depositing of the bar-weight by the smaller weight form a geometrical progression of common ratio $\frac{2M-p}{2M+p}$.

11. Two particles start simultaneously from rest at A to slide down two smooth tubes AB, AC, fixed in a vertical plane. Show that the line joining the particles at any instant makes with the line bisecting the angle BAC the same angle as the bisector makes with the horizon, and find the acceleration of one particle relative to the other.

12. A particle is held at rest close under a smooth rod which is inclined at an angle α to the horizon. When the particle is set free the rod begins to move with a horizontal acceleration f; show that if $f > g\cot\alpha$, the particle moves in a straight line with an acceleration $\sin\alpha \times \sqrt{g^2+f^2}$.

13. A string $OABC$ is trisected at A and B, and particles of equal mass are fastened to the points A, B, and C. If the end O is fastened to a fixed point on a horizontal plane, and if the string is straightened on the plane, and then set in motion so as to remain straight, prove that the tensions of the three portions of the string will be in the ratios 6 : 5 : 3.

14. A mass m on a smooth horizontal table is connected with a mass m' below it by a string of length c which passes through a smooth hole in the table. Show how to start them so that m may describe a circle with constant speed on the table, and m' move in a horizontal circle with constant speed, the time of revolution being the same for both. If the time of revolution is given, find the least value of c that the problem may be possible.

15. A flexible heavy string, length $2l$, is moving over a smooth fixed small pulley, the two unequal portions hanging vertically. Prove that at the instant when its middle point is at a distance x below the pulley, the acceleration with which it is moving is $f=\frac{x}{l}\cdot g$. Find also the tension of the string at any assigned point of the descending portion at the same instant.

16. A small pulley of mass M is lying on a smooth table; a light string passes round the pulley and has masses m and m' attached to its ends, the two portions of the string being perpendicular to the edge of the table and passing over it so that the masses hang vertically; prove that the pulley moves with acceleration

$$\frac{4mm'g}{M(m+m')+4mm'}.$$

17. A string passing round a smooth fixed pulley is attached at one end to a weight, and at the other to a smooth weightless pulley. A string passes round this weightless pulley, and is attached at one end to the ground and at the other to a weight equal in mass to the first mentioned weight. All the hanging portions of string are vertical. Prove that the first mentioned weight ascends with an acceleration equal to one-fifth of that due to gravity.

18. Two masses P and Q $(P > Q)$ are suspended by a light string over a pulley of inappreciable mass. After moving for a time t, a mass $P-Q$ is instantaneously attached to the ascending mass Q. Find the whole motion of the system.

Determine the circumstances of the motion if $P-Q$ were attached not to Q but to the string some way above Q, and show that the final velocity is $\frac{P-Q}{2P}gt$.

19. If to the smaller weight of an Atwood's machine is attached a string which, passing under a smooth pulley fixed vertically beneath on a table, is fastened to a weight (m') on the same table, the coefficient of friction being μ, find the tension of both strings if motion ensues.

20. Over a smooth light pulley is passed a string supporting at one end a weight of mass 6 lbs., and at the other end a small pulley of mass 1 lb. A string, with weights whose masses are 2 lbs. and 3 lbs., is passed over the second pulley; prove that the speed of the 2 lb. mass at the end of two seconds will be $\frac{26}{59}g$.

21. A particle of mass m is attached by a string to a fixed point C and by another string to a smooth ring of mass m' which can freely revolve round and slide along a vertical rod passing through C, both strings being weightless and inextensible. If the lengths of the vertical projections of the strings are x and y respectively when the whole system is revolving with constant angular velocity ω about the rod, prove that

$$\frac{m\omega^2}{g}=\frac{m+m'}{x}+\frac{m'}{y},$$

the ring being so small that it may be treated as a particle.

22. One end of an elastic string of natural length a is fastened to a point distant a below a smooth small fixed pulley; the string passes over the pulley and has fastened to the other end a weight of

mass m; prove that this weight when disturbed from its position of equilibrium performs harmonic oscillations about that position of period $2\pi\left(\frac{am}{\lambda}\right)^{\frac{1}{2}}$, where λ is the tension which the string would have if it were stretched to double its natural length.

23. A rough vertical circle carrying a bead turns in its own plane about its centre with uniform angular velocity greater than

$$\sqrt{\frac{g}{a}}\cdot\left(1+\frac{1}{\mu^2}\right)^{\frac{1}{4}},$$

where a is the radius, and μ the coefficient of friction. Prove that the bead will never slip.

80. Momentum of a System.

DEFINITION. *The vector sum of the momenta of a system of particles, estimated relative to a set of kinetic axes is called the* **linear momentum** *of the system.*

The linear momentum of a system resolved parallel to any line is the resolved part of this vector sum.

The word 'linear' is usually omitted, except when its omission would lead to ambiguity.

Thus if m_1, m_2, m_3, ... be the masses of the particles, $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, ... their velocities, u_1, u_2, u_3, ... the resolved parts of the velocities parallel to Ox, the momentum of the system is $\Sigma m\dot{\alpha}$, and the momentum resolved parallel to Ox is Σmu.

Internal and External Forces. In considering such a system of particles any part of the mass-acceleration of a particle of the system which is due to another particle of the system is called an *internal* force, any part of the mass-acceleration due to a particle outside the system is called an *external* force.

Thus *internal forces occur in equal and opposite pairs* (Third Law). Such a pair of forces is sometimes called a *Stress.*

In the two following propositions we consider the effect on the momentum of the system of (1) internal, (2) external forces.

81. PROPOSITION. *The momentum of a system is unaltered by any action, whether force or impulse, between the pairs of particles of the system.*

The forces between the particles may be grouped into stress pairs, *i.e.* if m_1, m_2 be the masses of any two particles of the system, and f_1, f_2 be their mutually induced accelerations,

$$m_1\bar{f}_1+m_2\bar{f}_2=0.$$

Hence the change of momentum arising in any finite time from

the stress between the particles m_1, m_2 is zero, and the same is true for each pair of particles, which proves the proposition.

As a particular case, the proposition is evidently true for impulses between pairs of particles.

82. PROPOSITION. *The rate of change of momentum of a system is equivalent to the vector sum of the external forces.*

With the notation of the last article, the momentum of the system is the vector sum

$$m_1\dot{\alpha}+m_2\dot{\beta}+m_3\dot{\gamma}+\ldots$$

and the rate of change of this is

$$m_1\ddot{\alpha}+m_2\ddot{\beta}+m_3\ddot{\gamma}+\ldots$$

Now $m_1\ddot{\alpha}$ is the resultant force on the particle m_1 (Law II.), and is therefore equivalent to the vector sum of the forces, internal and external, on that particle.

Hence $$m_1\ddot{\alpha}+m_2\ddot{\beta}+m_3\ddot{\gamma}+\ldots$$

is equivalent to the vector sum of the internal forces + the vector sum of the external forces on the system.

But the vector sum of the internal forces is *zero*, since they occur in stress pairs (Law III.).

∴ **The rate of change of momentum of the system is equivalent to the vector sum of the external forces.**

Corollary 1. The change of momentum of the system in a given time is equivalent to the vector sum of the impulses of the external forces for that time.

As a particular case, when the system is subject to instantaneous impulses, the change of momentum is equivalent to the vector sum of the external impulses.

Corollary 2. The resolved part of the rate of change of momentum of the system in any direction is equivalent to the algebraic sum of the resolved parts of the forces in that direction.

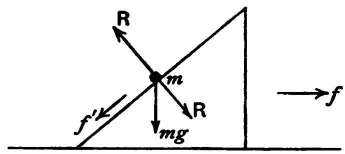
Corollary 3. (*a*) If the vector sum of the external forces is constant in direction, the rate of change of momentum of the system is entirely parallel to that direction, that is, the resolved momentum perpendicular to that direction remains unchanged.

(*b*) If the vector sum of the external forces is zero, the momentum of the system remains unchanged.

The statements (*a*) and (*b*) are known as the Principle of the **Conservation of Momentum.**

83. The following example, important in itself, will show the manner in which the above principle enables us to deal with simple cases of the motion of *rigid bodies.*

Example. *A particle of mass* m *slides down a wedge of mass* M, *which is capable of moving horizontally on a smooth table. Determine the motion.*



[The particle is supposed to be placed symmetrically on the wedge so that in the ensuing motion the latter does not rotate. M may for the present be regarded as the sum of the masses of the particles of the wedge; but see further, § 87 below.]

Let the acceleration of the wedge, which is horizontal, be f (left to right), and let the acceleration of the particle m *relative to the wedge,* and parallel to the line of greatest slope be f'. Then the horizontal acceleration of the particle relative to the table (*i.e.* the kinetic axes) is $f-f'\cos\alpha$ (left to right).

Now, the reactions of the table on the particles which compose the base of the wedge are all at right angles to the table, and consequently there is no external *horizontal* force on the system; the particles of the wedge are all moving with horizontal acceleration f, and therefore the rate of change of momentum of the system of particles composing the wedge, resolved horizontally, is Mf.

Thus for the system of wedge and particle

$$Mf+m(f-f'\cos\alpha)=0. \qquad \text{(i.)}$$

Further, resolving for the particle parallel to the line of greatest slope, we have

$$m(f'-f\cos\alpha)=mg\sin\alpha. \qquad \text{(ii.)}$$

From (i.) we have $\quad f'=\dfrac{M+m}{m\cos\alpha}\cdot f,$

and then (ii.) gives $\quad f'=\dfrac{(M+m)\sin\alpha}{M+m\sin^2\alpha}\cdot g,$

$$f=\frac{m\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\cdot g.$$

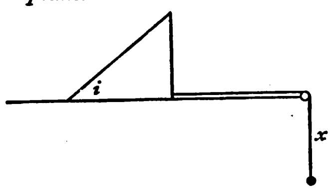
If we require the magnitude of the reaction R between the wedge and the particle, we note that the horizontal rate of change of momentum of the wedge must be equal to the horizontal resolved part of R, or

$$Mf = R \sin \alpha, \text{ which determines } R.$$

84. When the system whose motion we are considering consists of masses connected by a string which passes over a smooth pulley, it is often convenient to realise that the motion will be the same as if the whole system were placed in a straight line with the string tight, and each mass were acted on by an external force of the same magnitude as before, and in the line of the string. This is evident, since the tension is the same on each side of the pulley. For example, in the problem considered in § 71, Ex. 2, we may write: External force on the system $= (m - m')g$. Mass of system $= m + m'$. Therefore acceleration $= \dfrac{m - m'}{m + m'} \cdot g$.

The following examples will furnish other cases in point:

Example 1. *A wedge whose section is a right-angled triangle rests on a rough table with the hypotenuse inclined to the table at an angle* i. *A heavy smooth chain of length* a *is fastened to the wedge near the right angle, and passes over a smooth pulley at the edge of the table, so that the wedge is just on the point of motion when a length* b *of the chain hangs over the edge. A heavy smooth particle is then placed on the inclined face of the wedge and slides down it. Prove that the particle will separate from the wedge when the end of the chain has descended a distance equal to* $\left(\dfrac{b}{\mu} + a\right) \cot i$, *where* μ *is the coefficient of friction between the wedge and the table both before and during motion, the motion taking place in a vertical plane.*



Let M be the mass of the wedge, m that of the chain. The value of the friction is μMg, and the weight of a length b of the chain is $\dfrac{b}{a} \cdot mg$. Hence, for equilibrium, $\mu Mg = \dfrac{b}{a} mg$, or $M = \dfrac{bm}{a\mu}$.

Now when the particle leaves the wedge, it will be moving with a vertical acceleration g. Let f be the acceleration of the wedge just as the particle leaves it; then the resolved part of f perpendicular to the face of the wedge must be just equal to the resolved part of the particle's acceleration in this direction;

$$\therefore \quad f \sin i = g \cos i, \text{ or } f = g \cot i.$$

When the chain has this acceleration, let a length x be hanging over; then the external force producing this acceleration in a mass $M+m$, *i.e.* the sum of the masses of the system composed of the wedge and chain, is $\frac{x}{a} . mg - \mu Mg$, the reaction between the wedge and particle being zero.

$$\therefore \quad (M+m)g \cot i = \left(\frac{x}{a} . m - \mu M\right)g$$

$$= \frac{x-b}{a} . mg;$$

$\therefore$ the distance the chain has descended

$$= x - b = a\left(\frac{M}{m} + 1\right)\cot i$$

$$= \left(\frac{b}{\mu} + a\right)\cot i.$$

2. A boy hangs on to a light rope which passes over a smooth pulley and has attached to the other end a mass equal to that of the boy. Discuss the motion if the boy starts to climb (i.) with constant acceleration, (ii.) with constant speed. What difference will it make if in the latter case the counterpoising mass is held still while the boy starts?

3. A system of masses counterpoised on a rope passing over a smooth pulley is initially at rest. Prove that if mass *descends* on one side, mass will also *descend* on the other.

A bucket of water has a large piece of cork glued to its bottom inside, and the whole is attached to a light rope which passes over a smooth pulley, and is maintained in equilibrium by a counterpoising mass attached to the other end. If the water dissolves the glue, prove that when the cork begins to float upwards the counterpoise will move *downwards*.

Centre of Mass of a System.

85. DEFINITION. *The* **centre of mass** *of a system of particles of masses* $m_1, m_2, m_3, \ldots$ *is the mean point (or centroid) of the points marking the positions of the particles, for the multiples* $m_1, m_2, m_3 \ldots$.

Recapitulating the formulae of §§ 16, 33, 39, if $a_1, a_2, a_3, \ldots$ be the vectors from the origin to the particles, $x_1, x_2, x_3 \ldots$ the x coordinates of the particles, $u_1, u_2, u_3, \ldots$ their resolved velocities, $f_1, f_2, f_3, \ldots$ their resolved accelerations parallel to Ox, we have the following scheme for the displacement, velocity, and acceleration of the centre of mass relative to the origin :

	Displacement.	Velocity.	Acceleration.
Vector.	$\frac{\Sigma ma}{\Sigma m}$	$\frac{\Sigma m\dot{a}}{\Sigma m}$	$\frac{\Sigma m\ddot{a}}{\Sigma m}$
Resolved parallel to Ox.	$\frac{\Sigma mx}{\Sigma m}$	$\frac{\Sigma mu}{\Sigma m}$	$\frac{\Sigma mf}{\Sigma m}$

with similar formulae for the other axes.

Examples.

1. Prove that the paths described by each of two particles relative to the other or to their centre of mass are similar curves. (Similar curves are the limits of similar polygons.)

2. Taking the orbit of the Earth relative to the Sun to be roughly a circle of 93,000,000 miles radius, and the mass-ratio of the Sun to the Earth to be roughly equal to 330,000 : 1, find the radius of the orbit described by the Sun relative to their common centre of mass.

86. Velocity and Acceleration of the Centre of Mass.

We can now represent the momentum of the system as that of a single particle.

Let Γ be the vector from the origin to the centre of mass of the system, so that $\dot{\Gamma}$, $\ddot{\Gamma}$ are vectors representing the velocity and acceleration of this point.

Then, from the formulae of the last article,

$$\frac{\Sigma m\dot{a}}{\Sigma m} = \dot{\Gamma}, \text{ or } \Sigma m\dot{a} = \dot{\Gamma} \, . \, \Sigma m.$$

Hence the momentum of the system is equal to that of a particle of mass equal to the sum of the masses of the particles of the system, placed at the centre of mass and moving with it.

Hence also the velocity of the centre of mass is unaltered by internal forces or impulses.

So too we have $\Sigma m\ddot{a}=\ddot{\Gamma}\Sigma m$.

We have shown that $\Sigma m\ddot{a}=$ the vector sum of the external forces.

Hence the acceleration of the centre of mass is the same as that of a particle, mass Σm, acted on by a force equal to the vector sum of the external forces.

87. These conceptions are of great importance. In the case of a *rigid body* moving without rotation, the motion of the whole body is described when that of one point of it is described. We have now learnt how to determine the motion of one such point, and the description of the motion of the body is thus in this case complete.

Further, if we define the **mass of a rigid body** as the sum of the masses of its particles, we see in what manner the mass of the body, so defined, enters into the equations which determine the motion of translation of the body.

Even when the body is rotating these properties of the centre of mass teach us much about its motion. For instance, if a stick be thrown into the air, we know that its centre of mass will have a constant downward vertical acceleration g, and will therefore describe a parabola, while the stick rotates about the centre of mass.

Example. A straight smooth groove is cut in a horizontal table, and a straight slit is cut at the bottom of the groove. A string of length l, attached at one end to a particle of mass m resting in the groove, passes through the slit and supports a particle of mass m' at its other end. If this particle be held displaced in the vertical plane containing the groove, the string being straight, and then let go, show that the path of m' is part of an ellipse whose semiaxes are l, $lm/(m+m')$, the major axis being vertical.

[The path of the centre of mass is a vertical straight line.]

88. Before making our concluding remarks on Newton's axes it is convenient to enunciate the following theorem :

The motion of a system is referred to a given set of kinetic axes, and the only external forces are such as would produce in each particle, if free, a common acceleration $\overline{\mathrm{F}}$. *Then a set of axes in directions fixed relative to the former, but whose origin moves with acceleration* $\overline{\mathrm{F}}$, *if used as kinetic axes will determine correctly the internal forces of the system.*

Denote the first set of axes by (1), the second by (2). Let $\bar{f}$ be the acceleration of any particle (mass m) of the system referred

to axes (1). Let $\bar{R}$ be the resultant of the internal forces on the particle; then referring to axes (1)

$$m\bar{f}=\bar{R}+m\bar{F},$$

or

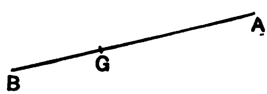
$$m(\bar{f}-\bar{F})=\bar{R}.$$

But $\bar{f}-\bar{F}$ is the acceleration of the particle referred to axes (2), which proves the proposition.

As an illustration, the Sun produces in the Earth and all bodies on its surface an acceleration of about $\frac{1}{1680}$ of the value of the acceleration due to gravity at the Earth's surface; as all these accelerations are in sensibly parallel directions and of equal magnitudes, they do not enter into calculations relative to Galileo's axes.

89. As a further illustration consider the following:

Example. *Two particles connected by a tight string are thrown whirling into the air. Determine the motion supposed to be in a vertical plane and the tension of the string.*



Let the particles be A and B, and their masses m and m'. Each mass would have, if free, the independent acceleration g. The centre of mass (G) has this acceleration (§ 86). It therefore moves in a parabola.

Further, to determine the tension of the string and the relative motion we may, since G has the acceleration g, choose it as origin, and axes through it parallel to Galileo's axes as kinetic axes, ignoring, of course, the forces mg, $m'g$ on the particles.

The motion is then, relative to G, uniform circular motion, so that the angular velocity remains constant, and equal to ω (say). The acceleration of A relative to G is $GA\,.\,\omega^2$.

Thus the tension of the string

$$=m\,.\,GA\,.\,\omega^2=\frac{mm'}{m+m'}\,.\,l\omega^2,$$

where l is the length of the string.

The student will find it instructive to solve the same problem by reference to Galileo's axes.

[θ being the inclination of the string to the vertical at any time, ω the angular velocity, $\dot{\omega}$ the angular acceleration of the string, the accelerations of m parallel and perpendicular to AB are

$$\frac{m'}{m+m'} \cdot l\omega^2 + g\cos\theta, \quad \frac{m'}{m+m'} \cdot l\dot{\omega} + g\sin\theta.$$

Then find the tension and prove that $\dot{\omega}=0$.]

90. The centre of mass of the solar system is, as remarked above, to be regarded as the origin of Newton's kinetic axes, the directions of the axes being determined by the fixed stars. The properties of the centre of mass just discussed are based ultimately on terrestrial experiments of the nature indicated in § 61 ; Newton's generalisation assumes that, as in the case of systems referred to terrestrial axes, so in the case of the solar system the motion of the centre of mass relative to bodies outside the system will be unaffected by the *internal* forces of the system. Moreover, it is a part of the Newtonian theory of Universal Gravitation that the particles of a limited material system will be affected by all distant bodies *each with the same acceleration*, relative to distant bodies, and the centre of mass will have this acceleration ; hence the internal forces of the system as calculated from the *relative motion* will form a unique and exhaustive system of stress-pairs. The direction in which the theory may be extended, if ever our knowledge becomes competent, is also obvious.

We may remark that although the motion of the bodies of the solar system relative to its centre of mass is at present only approximately known, so far as our knowledge goes theory and observation are at one. The mass-ratio of the Sun to even the most massive planet (Jupiter) is so great (1000 : 1) that to a first approximation the Sun's centre may be used instead of the centre of mass for origin, and *is* so used in the elementary theory of the motions of the solar system given in books on "The Dynamics of a Particle."

91. Theorem of Moments.

The moment of a localised vector about a point, or about an axis, has already been defined. When the moment of the momentum or of the mass-acceleration of a particle, or of the force acting on a particle, is spoken of, it is understood that these vectors are to be localised at the point which indicates the instantaneous position of the particle.

PROPOSITION. *A system of particles is moving in one plane ; to prove that the algebraic sum of the moments of their mass-accelera-*

tions about any "fixed" point in the plane is equal to the sum of the moments of the external forces about that point.

If m be the mass of one particle, $\bar{f}$ its acceleration, $m\bar{f}$ is equal to the vector sum of the forces, internal and external, on the mass m.

Hence (§ 17, Chapter I), the moment of $m\bar{f}$ about the given point = sum of moments of internal and external forces acting on the particle m.

The same is true for each particle of the system.

Therefore the sum of the moments of the mass-accelerations about the given point = sum of moments of forces, internal and external, on each particle.

But the sum of the moments of the internal forces is zero, since they can be grouped into stress-pairs. Therefore the sum of the moments of the mass-accelerations = the sum of the moments of the external forces.

Corollary 1. If the motion of the particles is not confined to one plane, a corresponding theorem is true for moments about an axis. (See § 17, Prop., Cor. 2.)

Corollary 2. A similar theorem holds for impulses, viz.

If a system of particles moving in one plane is acted on by impulses in that plane, the sum of the moments of the momenta generated, about a fixed point in the plane, is equal to the sum of the moments of the external impulses about that point; and mutatis mutandis *the theorem will hold for moments about an axis.*

92. From the theorem of moments and the theorem of § 86 which states that the centre of mass of a system moves like a particle subject to the vector sum of the external forces, follow at once the rules for the equilibrium of a rigid body under the action of external forces applied to its particles. Assuming that the reader is familiar with these rules, we need here only note their place in the logical development of the subject.

In particular we may assume in future that the sum of the moments of the weights of the particles of a rigid body about any horizontal axis is equal to the moment of the sum of the weights localised so as to pass through the centre of mass.

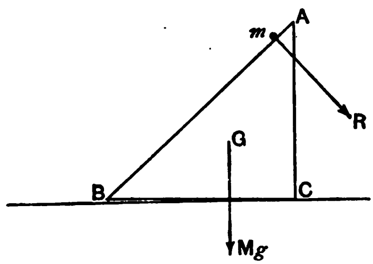
It will be useful, further, to draw the reader's attention to the case of a rigid rod of negligible mass, to which particles of given masses are attached.

If the reactions of the particles on the rod are *forces*, and every point of the rod is moving with finite acceleration, these reactions must satisfy the conditions that their vector sum is

zero, and the sum of their moments about any fixed point is zero. If the reactions are impulses, and the *change of velocity* of every point of the rod is finite, the impulses satisfy similar conditions.

93. *The student should especially note that he may in every case equate the sum of the moments of the mass-accelerations about any fixed axis to those of the external forces about that axis.* It may happen that certain particles of the system will, at the instant at which moments are taken, lie on this axis, and in that case the mass-accelerations of the particles, as also any forces applied to them, will have no moment about the fixed axis. An instance of this is found in the following example:

Example 1. *A rectangular wedge* ABC *of mass* M *rests on a smooth horizontal table with the face* AB *opposite the right angle inclined at an angle* a *to the horizon. A smooth particle of mass* m *is placed symmetrically at the top of the wedge, and allowed to run down the inclined face. Prove that, if in the ensuing motion there is no tilting of the wedge,* M *must be greater than* $m \sin^2 a$.



If R be the reaction of the particle on the wedge, f the acceleration of the wedge supposed not to tilt, then, as in the Example of § 83,

$$Mf = R \sin a, \quad f = \frac{m \sin a \cos a}{M + m \sin^2 a} \cdot g,$$

and therefore
$$R = \frac{Mmg \cos a}{M + m \sin^2 a}.$$

Suppose the wedge, when the mass m is placed at the top of it, to be *just about to tilt*; then the reactions of the plane on the particles of the base all pass through the edge C.

We shall take moments about the fixed straight line in the plane with which the edge C *instantaneously coincides.*

If h be the height of the wedge, the moment of R, when m is at the top, is $Rh \sin a$, and the moment of the weights of the particles composing the wedge = moment of Mg supposed to pass through the centre of mass

$$=Mg \cdot \tfrac{1}{3}h \cot a.$$

Similarly, the sum of the moments of the mass-accelerations of the particles of which the wedge is composed is $Mf \cdot \tfrac{1}{3}h$.

We have thus $Rh \sin a - \tfrac{1}{3}Mgh \cot a = \tfrac{1}{3}hMf$.

Now if the wedge is *not* about to tilt, the terms on the left-hand side of this equation would have to be numerically increased by the moments of the reactions of the plane on the particles of the base of the wedge, so that the condition that the wedge should not tilt initially is

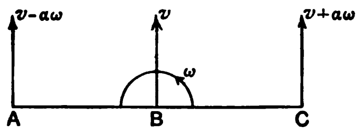
$$R \sin a - \tfrac{1}{3}Mg \cot a < \tfrac{1}{3}Mf,$$

which, substituting for R and f, gives

$$M > m \sin^2 a.$$

It is evident that if the wedge does not tilt at once it will not do so at all, since the moment of the external forces about the edge C is greater at the beginning of the motion than at any subsequent time.

Example 2. *Three particles of equal masses attached to the ends and the middle point of a rigid weightless rod lie on a smooth table. If an end particle receive an impulse in the direction perpendicular to the rod, prove that the particles will start off with speeds in the ratio of* 5 : 2 : 1.



Let A, B, C denote the particles, let m be the mass of any one, and let $AB = BC = a$.

All the particles will begin to move at right angles to the initial position of the rod.

Let v be the speed with which B begins to move, ω the initial angular velocity of the rod. Then the velocities of A, C relative to B are of magnitude, $-a\omega$, and $+a\omega$ respectively (§ 45, Cor.).

Let C be the particle to which the impulse is given. Take moments about the point of the table *with which* C *instantaneously coincides.* The impulse has no moment, nor has the

momentum of the particle C. The momenta *generated* in the other two particles are respectively $m(v-a\omega)$ and mv.

$$\therefore\quad 2am(v-a\omega)+a\,.\,mv=0,$$

or
$$3v=2a\omega,\text{ or } a\omega=\frac{3v}{2}.$$

The speeds of the particles are therefore

$$-\frac{v}{2},\ v,\text{ and }\frac{5v}{2}\text{ respectively.}$$

The student should consider what the subsequent motion of the system will be.

3. Prove that the tension of a light string is unaltered by passing round a smooth circular pulley.

4. A block with a plane base rests on a smooth horizontal plane. A horizontal impulse is applied to it. Determine the condition that the impulse shall not tilt the block.

5. If the impulse is not horizontal, but inclined downwards to the plane, prove that the condition is that the line of action of the impulse should cut a horizontal plane through the centre of mass in a point which lies vertically above some point of the area enclosed by a tight string drawn round the base.

6. A railway carriage is travelling round a curve of varying curvature, the dimensions of the carriage being small when compared with the radius of curvature of the path of any point of it.

Prove that if the radius of curvature of the curve at any point be $<\dfrac{2v^2h}{ag}$, where h is the height of the centre of mass of the carriage above the rails, a is the breadth of the track, and v is the speed, the carriage cannot pass this point without the inner wheels leaving the line.

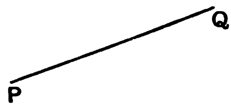
What are the horizontal and vertical pressures on the outer rail when this is just about to happen, and what is the direction of the resultant pressure?

94. The moment of the momentum of a particle moving in a plane, about any "fixed" point in the plane, is called the *angular momentum* of the particle about that point.

From § 55 we see that the *rate of change of angular momentum of a particle* about a fixed point is equal to the moment of its mass-acceleration about that point. If then the resultant force on the particle always passes through a fixed point, the mass-acceleration, which is its expression, has *no* moment about the

point, and the *angular momentum of the particle about the point is constant.* This is the theorem of the conservation of angular momentum for a particle.

95. Initial Motions. Suppose that a system of particles is in equilibrium under given constraints and that one of the constraints is suddenly removed, we may often by elementary methods determine the initial accelerations of the particles and the initial values of the forces acting on them. The problem is simplified by the fact that *all initial velocities and angular velocities are zero.* Thus the acceleration of a particle Q

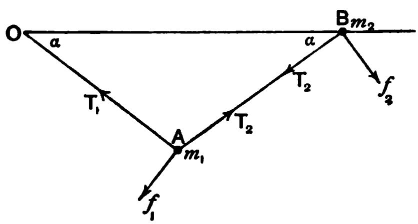


relative to a particle P to which it is attached by a light inextensible string in general consists of (i.) an acceleration J perpendicular to PQ, and (ii.) an acceleration $\omega^2 . PQ$ parallel to QP (see § 49); but in an initial motion the latter is zero.

Example. *A light string fixed at one end has two particles attached to it,* m_1 *at its middle point,* m_2 *at its free end.* m_2 *is held in the same horizontal line as the fixed end, the two segments of the string making an angle* a *with the horizon. Prove that if* m_2 *is let go the initial tension of the string at the fixed end is*

$$m_1(m_1+m_2)g \sin a/(m_1+m_2 \sin^2 2a),$$

provided $a < \frac{\pi}{4}$. *What happens if* $a > \frac{\pi}{4}$?



Let O be the fixed point, OA, AB the two portions of the string, and let the initial tension of OA be T_1, that of AB, T_2.

Since the angular velocities of OA, AB are both initially zero the acceleration of A relative to O is initially perpendicular to

OA, while that of B relative to A is initially perpendicular to BA. Let these be f_1, f_2 respectively.

Then, relative to the kinetic axes, the accelerations of m_1 resolved horizontally and vertically are $-f_1 \sin \alpha$ (left to right), $f_1 \cos \alpha$ downwards; while those of m_2 are $-f_1 \sin \alpha + f_2 \sin \alpha$ (left to right), $f_1 \cos \alpha + f_2 \cos \alpha$ downwards.

Hence resolving horizontally and vertically for each particle,

$$-m_1 f_1 \sin \alpha = (T_2 - T_1) \cos \alpha, \quad m_1 f_1 \cos \alpha = m_1 g - (T_1 + T_2) \sin \alpha,$$

$$m_2(-f_1 + f_2) \sin \alpha = -T_2 \cos \alpha, \quad m_2(f_1 + f_2) \cos \alpha = m_2 g + T_2 \sin \alpha.$$

These four equations determine the four unknowns, f_1, f_2, T_1, T_2. Eliminating f_1, f_2, T_2, we obtain at once the required result.

If $\alpha = \dfrac{\pi}{4}$, it will be found that $T_2 = 0$, and $f_1 = f_2 = \frac{1}{2} g \sec \alpha$; from which it follows that the string AB at once becomes slack, and that m_2 descends freely with acceleration g. The same will be the case if $\alpha > \dfrac{\pi}{4}$.

96. In another class of problems in which the particles are acted on by impulses, and so move off with finite velocities, it is often possible to determine the initial radius of curvature (R) of the path of a particle which moves off with velocity v from the fact that the normal acceleration is $\dfrac{v^2}{R}$.

Example. *Two particles of masses* p *and* q *are connected by a string which is straight and passes through a smooth fixed ring. The whole is on a smooth horizontal table, and the particles are projected at right angles to the string. Prove that the initial curvatures of their paths are*

$$\mathrm{q}(\mathrm{p}+\mathrm{q})^{-1}(\mathrm{bu}^2+\mathrm{av}^2)/\mathrm{abu}^2 \quad \text{and} \quad \mathrm{p}(\mathrm{p}+\mathrm{q})^{-1}(\mathrm{bu}^2+\mathrm{av}^2)/\mathrm{abv}^2$$

respectively, where u, v *are the initial velocities,* a *and* b *the portions into which the string is initially divided.*

Let A, B be the particles, f the acceleration with which the point of the string initially in contact with the ring begins to slip, supposed positive from left to right.

The acceleration of A relative to this point is $-\dfrac{u^2}{a}$.

Therefore the acceleration of A (left to right) is $f-\frac{u^2}{a}$.

Similarly the acceleration of B (left to right) is $f+\frac{v^2}{b}$.

Hence by the Third Law,

$$p\left(f-\frac{u^2}{a}\right)+q\left(f+\frac{v^2}{b}\right)=0,$$

$$\therefore\ f=(p+q)^{-1}\left(\frac{pu^2}{a}-\frac{qv^2}{b}\right).$$

And acceleration of $A=f-\frac{u^2}{a}=-q(p+q)^{-1}\frac{av^2+bu^2}{ab}$.

Now let R be the initial radius of curvature of A's path. The value of this acceleration (left to right) must be also $-\frac{u^2}{R}$.

Whence $\quad \frac{1}{R}=q(p+q)^{-1}(av^2+bu^2)/abu^2$.

Examples.

1. A particle of mass m is attached to a string which is attached to a fixed point, and drawn aside from the vertical through an angle α, and suddenly released from rest. Prove that the initial value of the tension is $mg\cos\alpha$, and the initial acceleration is numerically equal to $g\sin\alpha$.

2. A string ABC has the point A fixed, and to B and C two particles of equal mass are attached. The system is drawn aside from the vertical and held at rest. Prove that if it be suddenly released the tensions of the portions AB, BC of the string are respectively

$$mg\frac{2\sin\alpha_1}{1+\sin^2(\alpha_1-\alpha_2)},\quad mg\frac{\sin\alpha_1\cos(\alpha_1-\alpha_2)}{1+\sin^2(\alpha_1-\alpha_2)},$$

where α_1, α_2 are respectively the inclinations of AB, BC to the horizon.

3. A particle of mass m is suspended from a fixed point by a string of length a, and from m is suspended another particle of mass m' by a string of length b. If a horizontal velocity be suddenly communicated to m, show that the tensions of the strings are immediately increased by amounts which are in the ratio

$$1+\frac{mb}{m'(a+b)}:1.$$

4. Three holes A, B, C in a smooth table form an equilateral triangle. Through these holes three smooth strings are passed supporting equal weights, the mass of each of which is equal to m,

and the other ends of the strings are all fastened to a mass M at the centre of gravity of ABC. If one of the strings is cut, show that the initial acceleration of M is $\frac{2mg}{2M+m}$.

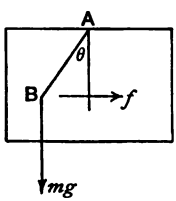
5. Three equal rings rest on a smooth vertical circular wire at the corners of an equilateral triangle of which one side is vertical, the uppermost being connected with the other two by means of strings. Find their tensions, and show that if the vertical string be cut the tension in the remaining string is instantaneously diminished in the ratio 3 : 4.

6. Two particles of masses m, m' lie on a smooth horizontal table connected by an inelastic string of length a. m is projected in the plane at right angles to the string. Prove that the initial radius of curvature of its path is $\frac{m+m'}{m'} \cdot a$.

97. *Motion relative to axes which retain constant directions while their origin moves with a given acceleration relative to the kinetic axes.*

As a simple case let us consider motion relative to a railway-carriage which is moving along a horizontal line with constant acceleration f relative to Galileo's axes.

A free particle falling in the carriage will, neglecting the resistance of the air, have relative to the carriage an acceleration equal to the vector difference of $\bar{g}$ and $\bar{f}$. In other words, its relative acceleration will be of magnitude $\sqrt{g^2+f^2}$, and inclined at an angle $\tan^{-1}\frac{f}{g}$ to the vertical. If the particle be dropped from the hand of a person in the carriage its velocity relative to the carriage is initially zero, and its path relative to the carriage will be a straight line.



Next consider the case of a particle suspended by a string AB from the roof, resting in relative equilibrium at an angle θ to the vertical. Call the tension of the string T, the mass of the particle m. Since the acceleration of the particle relative to the kinetic axes is f, we have, resolving horizontally and vertically,

$$T\sin\theta = mf, \quad T\cos\theta - mg = 0,$$

whence $$T = m\sqrt{f^2+g^2}, \quad \tan\theta = \frac{f}{g}.$$

Now these and similar cases may be described by saying that the particle behaves *relative to the carriage* as if axes fixed in

the carriage were kinetic axes, and the acceleration due to gravity were replaced by an acceleration equal to the vector difference of $\bar{g}$ and $\bar{f}$. Let us call the force represented by the product of the mass of the particle and the vector difference of $\bar{g}$ and $\bar{f}$ the *apparent weight* of the particle; then it is important to notice that the values as calculated from the relative motion of all forces acting on the particle except the apparent weight are the true values of these forces as calculated from the motion relative to the true kinetic axes, by which alone force can be defined; in fact the term mf only changes sides of the equation.

This method of description is sometimes convenient; but the student is advised whenever he makes use of it to interpret his equations relative to the kinetic axes.

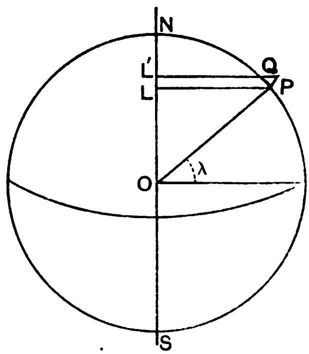
Examples.

1. How must the particle, when suspended from the roof and drawn aside from the vertical plane containing the string in the position of relative equilibrium, be started, that it may appear to an observer in the carriage to move in a plane?

2. If the string be of length l, how, and with what speed, must the particle be started in order that it may appear to the observer to describe a circle of radius a with constant speed?

98. *Effect of the Earth's Rotation on the Acceleration due to Gravity.*

For questions in which the rotation of the Earth is taken into account, Galileo's axes are, of course, inadequate. We shall, in the discussion which follows, assume that the origin of the kinetic axes is to be the centre of the Earth, the directions being determined by lines drawn thence to the fixed stars. The situation of an observer on the Earth's surface relative to these axes is nearly analogous to that of the observer in the railway carriage just discussed, relative to Galileo's axes.



We shall also assume that the Earth is a sphere. Let NS represent the polar axis, O the centre of the sphere, OP a radius to a given point P on the surface, λ the inclination of OP to the

plane of the equator, NPS the meridian through P, PL the perpendicular from P on the polar axis, ω the Earth's angular velocity relative to the stars, R the Earth's radius.

Then relative to the kinetic axes (origin O) P is describing a circle with constant speed $\omega \,.\, PL$. Its acceleration is therefore $\omega^2 \,.\, PL$ or $\omega^2 R \cos\lambda$ in the direction of PL. Moreover, any point Q near P and rigidly attached to the earth will have an acceleration $\omega^2 QL'$ (where QL' is the perpendicular from Q on NS) which is very approximately the same as $\omega^2 PL$.

In fact, the vector difference of the accelerations of P and Q

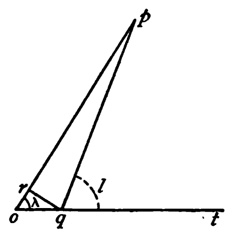
$$\begin{aligned} &= \omega^2 \overline{QL'} - \omega^2 \,.\, \overline{PL} \\ &= \omega^2 (\overline{QL'} - \overline{PL}) \\ &= \omega^2 (\overline{QP} + \overline{PL} + \overline{LL'} - \overline{PL}) \\ &= \omega^2 (\overline{QP} + \overline{LL'})\,; \end{aligned}$$

and this latter is a vector whose magnitude is very small compared with that of $\omega^2 \,.\, \overline{PL}$.

Neglecting the very small difference between the accelerations of P and Q, we may regard every point near P and fixed with regard to Galileo's axes (origin P) as moving with an acceleration, relative to the kinetic axes through O, which is the same for all the points and is sensibly constant during the intervals occupied by most motions of bodies near the Earth's surface.

If then G be the value of the acceleration due to gravity at P, relative to the kinetic axes through O, g its apparent value at the Earth's surface at P, we at once see from the principle of relative accelerations that $\bar{g}$ is the vector difference of $\bar{G}$ and $\overline{\omega^2 R \cos\lambda}$.

The direction of $\bar{G}$ is, by the theory of universal gravitation, PO. The direction of the apparent acceleration due to gravity is of course the *vertical*, defined as the direction taken by the plumb-line, or as the normal to the surface of a liquid at rest. The latter method is used for determining the vertical in Astronomy, and the latitude of the place (l) is defined as the inclination of the vertical so determined to the plane of the equator.



If then, po, pq, qo represent the accelerations $\bar{G}$, $\bar{g}$, $\overline{\omega^2 R \cos\lambda}$, respectively, and we produce oq to t, angle $poq = \lambda$, and angle $pqt = l$ the latitude.

Now, taking R equal approximately to 4000 miles, and neglecting the difference between the lengths of a sidereal and a mean solar day, so that $\omega = \frac{2\pi}{24 \times 60 \times 60}$, we obtain

$$\omega^2 R = \cdot 1117 \text{ f.s.s.}$$
$$= \tfrac{1}{9} \text{ f.s.s. nearly.}$$

[If R = Earth's equatorial radius = 20,923,600 feet, and we take ω equal to $\frac{2\pi}{86164}$ radians per second, its true value, we obtain $\omega^2 R = \cdot 11126$ f.s.s.]

Now g is equal to 32·2 approximately in the latitude of London, and varies from 32·088 at the equator to (probably) 32·253 at the poles; consequently the ratio $\frac{\omega^2 R}{g}$ is very nearly equal to $\frac{1}{289}$ in all latitudes.

It follows that the angle opq is very small, and drawing qr perpendicular to op we have

$$G - g = ro = oq \cos\lambda = \omega^2 R \cos^2\lambda,$$

or
$$g = G - \omega^2 R \cos^2\lambda,$$

or
$$= G - \omega^2 R \cos^2 l,$$

writing l for λ in the small term $\omega^2 R \cos^2\lambda$.

Further, the circular measure of the angle $l - \lambda$ is approximately equal to

$$\frac{rq}{pq}, \text{ or } \frac{\omega^2 R \sin\lambda \cos\lambda}{g},$$

for which we may write

$$\frac{\omega^2 R \sin l \cos l}{g}.$$

Thus it appears that the effect of the Earth's rotation is to diminish the acceleration of a falling body near the Earth's surface in latitude l by $\omega^2 R \cos^2 l$, and to turn its direction through an angle whose circular measure is

$$\frac{\omega^2 R \cos l \sin l}{g}.$$

In the latitude of London the values of these quantities are respectively about ·043 f.s.s. and 5′ 47″.

99. We see further that the quantity mg which we are accustomed to call the weight of a mass m is a quantity analogous to the "apparent weight" of the particle as discussed in § 97; and further that any forces acting on the particle, except its weight, will so far as our approximation goes be correctly determined relative to kinetic axes through the centre of the Earth from the motion relative to Galileo's axes. For instance, if the particle is suspended in relative equilibrium by a string the tension of the string is mg even when we refer to the axes through the Earth's centre.

Examples.

1. Prove that a column of sufficient height erected at the equator will cease to have any weight.

2. Assuming that at the equator the effect of the Earth's rotation is to diminish the apparent weight of a body by $\frac{1}{289}$ part, find what would be the length of the day if bodies at the equator had no weight.

Also in this case find the direction of the plumb-line in any other latitude.

3. "A railway train travelling west is heavier than when travelling east." What kinetic axes does such a statement imply? Find the difference in weight for a train weighing 180 tons, and travelling 60 miles an hour, in latitude 60°.

4. A heavy particle is attached to one end of an inextensible string, the other end of which is moving with constant acceleration f inclined at an angle a to the horizontal; prove that the particle, if properly started, will move in a straight line with acceleration f, with the string inclined to the vertical at an angle $\cot^{-1}\dfrac{g+f\sin a}{f\cos a}$.

5. Prove that if a stone is let fall from relative rest at a height of 100 feet at the equator, it will, neglecting the resistance of the air, strike the ground at a distance of ·018 foot to the east of the vertical through its initial position.

6. Assuming the Earth to be a sphere, radius 4000 miles, prove that the number of seconds in the angle between the vertical and the plumb line in latitude 15°, taking g as 32, is $n \cdot \left(\dfrac{5}{48}\right)^2 \cdot \pi$ nearly, where n is the number of feet in a mile.

Examples on Chapter III.

1. What kinetic axes, if any, can be chosen so as to give a meaning to the following expressions or statements: "the mass of the Earth"; "action and reaction are equal and opposite, and

therefore the weight of a body resting on the ground is equal and opposite to the pressure of the ground on the body"; "the force exerted by the Moon on the Earth"; "the force exerted by a star on the Earth"; "the weight of the Sun" (a phrase often occurring in Elementary books on Astronomy)?

2. A smooth straight rod is constrained to be a tangent to a fixed parabola; prove that, if it turn with angular velocity proportional to the cube of the sine of its inclination to the axis, a bead on it can be projected so as to describe the parabola, its velocity relative to the rod remaining constant; and that this velocity will not be constant in magnitude if the angular velocity follows any other law.

3. A leaky vessel, originally full of water, is projected upwards (vertically) with velocity V. Discuss the question whether the opportunity of leakage would affect the motion of the vessel.

4. Two platforms A and B of equal mass m, each carrying an animal of mass $km (k > 1)$ start with different speeds from alongside, and are constrained to move along parallel straight lines; the one, A, which at first travels fastest, is retarded by a constant force. Prove that, after T seconds, when B overtakes A, the relative velocity of the platforms is equal to their relative velocity when starting. Also, if at the instant of passing, the two animals jump across at right angles to the parallel lines and exchange platforms, prove that the platforms will again pass each other after $\frac{k-1}{k+1} . T$ seconds.

5. A (mass m) and B (mass m') are two particles connected together by a stretched string of length $2a$ lying on a smooth table so that AB is perpendicular to its edge, and A is beginning to fall off. Prove that in time $\frac{\pi+4}{2}\sqrt{\frac{a}{g} . \frac{m+m'}{m}}$ the particles will have fallen the same distance.

6. Two equal particles, each of mass m, connected by a light inelastic string of length l rest on a smooth horizontal table, and a blow I is given to one of the masses in a direction at right angles to the string. Determine the tension of the string, and show that each particle describes a series of cycloids; find the time of describing a complete cycloid. Draw a figure illustrating the motion.

7. Two equal particles connected by a weightless rod describe horizontal circles on the internal surface of a fixed sphere, so that the plane through the rod and the centre is vertical and turns uniformly. Find the angular velocity of this plane in terms of the angles α, β which the radii of the horizontal circles subtend at the centre of the sphere, and prove that the thrust in the rod is to the weight of either particle as

$$\sin\alpha\sin\beta\tan\tfrac{1}{2}(\alpha-\beta) : \cos\tfrac{1}{2}(\alpha+\beta)\cos(\alpha-\beta).$$

8. Four particles A, B, C, D lie on a smooth table at the corners of a rhombus. The particles are connected by four light inextensible strings AB, BC, CD, DA; the mass of A is m_1, the masses of B and D are m_2 respectively, that of C is m_3. The angle α at A is acute. A blow is given to A along the diagonal, away from C. Prove that the ratio of the initial velocity of C to that of A is

$$m_2 \cos\alpha : m_2 + m_3(1 - \cos\alpha).$$

9. Two particles, each of mass m, are moving with equal velocities at right angles to the line joining them; they are connected by a loose string, to the middle point of which a second string is attached, at the other end of which is a particle of mass M, which is at rest in the plane of motion and equidistant from the particles m; prove that when the strings tighten the direction of motion of each of the particles m will be suddenly deflected through an angle

$$\tan^{-1}\frac{M \sin\alpha}{(2m - M)\cos\alpha + 2m + M}$$

when α is the angle between the two portions of the first string at the moment of tightening, and the masses of the strings are neglected.

10. A long smooth straight wire is inclined at an angle α to the horizontal, and a smooth ring is instantaneously at rest on it. If the wire be suddenly moved horizontally (without rotation) in the vertical plane passing through it with velocity u, prove that the path of the ring in space is a parabola, and find its latus rectum.

11. A particle of mass m is tied by a string of length l to the vertex of a right circular cone of mass M and vertical angle 2α, placed on its base on a perfectly rough horizontal table. Prove that the greatest angular velocity with which the particle can be projected so as to describe a horizontal circle without upsetting the cone is

$$\left(\frac{g}{l}\right)^{\frac{1}{2}}\left[1 + \left(1 + \frac{M}{m}\right)^2 \tan^2\alpha\right]^{\frac{1}{4}}.$$

12. A wheel and axle is suspended by the string which surrounds the axle. From its axis hangs a mass M, and the string which surrounds the wheel supports another mass m. Assuming the wheel and axle and the strings to be massless, and supposing the wheel and axle to be falling downwards, show that its acceleration is

$$\frac{Mb^2 - m(a-b)b}{Mb^2 + m(a-b)^2}g,$$

where a is the radius of the wheel, b that of the axle.

13. A horizontal tube, closed at both ends, has in it a ball of cork, and the remainder is filled with water. If the tube be rotated about one end, in which direction will the ball travel?

14. A rough horizontal table rotates about a vertical axis; two masses connected by a string lie on the table so that the line of the string produced passes through the axis of rotation. Show that, when the more distant mass rests as far as possible from the axis of rotation, the distances are such that a third free mass placed at the distance of the centre of gravity of the first two masses from the axis would just not slide.

15. A window is supported by two cords passing over pulleys in the framework of the window (which it loosely fits), and is connected with counterpoises each equal to half the weight of the window. One cord breaks, and the window descends with acceleration f. Deduce that the coefficient of friction between the window and the framework is

$$\frac{(g-3f)a}{(g+f)b},$$

when a is the height and b the breadth of the window.

16. A wedge of mass M and angle α is held at rest with one face in contact with a smooth table. A string ABC passes over a smooth pulley at B, the highest point of the wedge, and supports a particle C of mass m in contact with the wedge. The other end of the string is fastened to a fixed point A in the same horizontal line as B. A and C being on opposite sides of the vertical through B. Prove that when the wedge is allowed to slide the tension of the string is suddenly diminished in the ratio

$$M+2m\sin^2\frac{\alpha}{2} : M+4m\sin^2\frac{\alpha}{2}.$$

17. A wedge of mass M and angle α rests on a smooth horizontal table with its edge parallel to an edge of the table, and is free to slide in the direction perpendicular to its edge. A string tied to the middle point of the edge passes over the edge of the table and supports a mass m. A particle is placed symmetrically on the slope of the wedge, and when the system is allowed to move from rest it is found that the particle, relatively to the wedge, does not move. Show that its mass is

$$m\cot\alpha - m - M.$$

18. A smooth wedge whose base angles are α and β, and whose mass is M, is placed on a smooth horizontal table, and two masses m and m' are attached to the ends of a string which passes over a smooth pulley at the summit. Show that the acceleration with which the wedge moves is

$$\frac{(m\sin\alpha \sim m'\sin\beta)(m\cos\alpha+m'\cos\beta)}{(m+m')(M+m+m')-(m\cos\alpha+m'\cos\beta)^2}\cdot g.$$

19. A body of mass m_1 has a plane face resting upon a wedge of mass m_2 and angle β, with which its angle of friction is ϵ_1. The wedge is placed upon a plane making an angle α ($<\beta$) with the

horizon, with the thick end down the plane and the thin edge horizontal; the angle of friction between the wedge and the plane is ϵ_2. If $\epsilon_1 < \beta - \alpha$, show that on the release of the system the body will slide with constant acceleration down the wedge, but the wedge will not slip on the plane provided that

$$(m_1 + m_2) \cos \epsilon_1 \sin (\epsilon_2 - \alpha) > m_1 \sin (\beta - \alpha - \epsilon_1) \cos (\epsilon_2 - \beta).$$

If ϵ_2 exceeds α, but is not large enough to satisfy the above condition, determine what will happen.

20. A light string of length l equal to that of a smooth fixed inclined plane of angle α passes round a pulley at the top of it, masses M, m which rest on the plane being attached to its ends; initially the masses are at rest, M at the top and m at the bottom of the plane; prove that, if motion takes place under the action of gravity, and an inelastic direct impact occurs half-way down the plane, the string will break if it cannot sustain an impulse equal to

$$\tfrac{1}{2}(M+m)^{-\frac{1}{2}}(M-m)^{\frac{3}{2}}(lg \sin \alpha)^{\frac{1}{2}};$$

and find the common speed of the masses after impact in terms of the impulse which will just break the string.

21. Two bricks of length $2a$ and of equal mass are placed symmetrically one above the other on a rough horizontal plane. A light string attached to the lower brick passes in the direction of the length of the bricks over a light pulley, and is tied to another equal brick hanging vertically. The coefficient of friction between the two bricks is μ, and is the same as that between the lower brick and the plane. Show that the two bricks will not move as one body unless $\mu > \frac{1}{5}$, and that, if μ has a value less than this, the upper brick will fall off after a time $2\sqrt{\dfrac{a}{(1-5\mu)g}}$.

22. Two smooth fixed planes make angles α, β with the horizon, and their intersections with any horizontal plane are mutually inclined at an angle γ. Two particles of masses m and m' lie respectively on the planes, and are connected by a light string passing over their common edge. Prove that a motion starting from rest with the acceleration of each particle constant is possible and that the tension of the string in this case is

$$\frac{mm'g}{m+m'} \cdot \frac{\sin \alpha \cot \beta + \sin \beta \cot \alpha + \cos \gamma (\cos \alpha + \cos \beta)}{\sqrt{\{\sin^2 \gamma + \cot^2 \alpha + \cot^2 \beta + 2 \cos \gamma \cot \alpha \cot \beta\}}}.$$

23. Two particles of masses m and m' are connected by a string which lies symmetrically on the faces of a smooth isosceles wedge (mass M), the string passing perpendicularly over the edge; the wedge is placed with its edge horizontal on a smooth inclined plane

of inclination θ, and the system is then let go; show that the wedge will move down the plane with acceleration

$$g\left[\sin\theta + \frac{(m-m')\sin\alpha\cos\alpha}{M+(m+m')\sin^2\alpha}\cdot\cos\theta\right],$$

where α is each of the angles at the base of the wedge and $\theta < \frac{\pi}{2} - \alpha$.

24. A wedge of angle α and mass M_1 moves on a smooth horizontal plane, and another wedge of the same angle and mass M_2 is placed on its inclined face so that the upper surface of the second wedge is horizontal. Show that, if a particle of mass m be placed on the upper surface which is smooth, its acceleration relative to the surface is to the acceleration of the lower wedge as $M_1 : M_2$, and is equal to

$$\frac{M_1(M_2+m)g}{(M_1+M_2)(M_2+m)\tan\alpha + M_1M_2\cot(\alpha-\lambda)},$$

where λ is the angle of friction between the inclined faces of the two wedges, the whole motion taking place in a vertical plane.

25. A particle lies on a horizontal table at the foot of a smooth wedge of angle α and height h, and the wedge is made to move along the table with constant acceleration f. Prove that if $f > g\tan\alpha$ the particle will ascend the plane. Show also that, if the wedge moves thus for a time t and then moves with constant velocity equal to that gained, the particle will just reach the top if

$$t = \left\{\frac{2gh\sec\alpha}{f(f\cos\alpha - g\sin\alpha)}\right\}^{\frac{1}{2}}.$$

26. A smooth wire in the form of the perimeter of an equilateral triangle is fixed in a vertical plane, and on each side a bead slides, the three beads being connected by a string which passes round smooth pulleys at the angular points of the triangle; find the common acceleration of the beads in terms of their masses m_1, m_2, m_3, and of the angle which one side of the triangle makes with the vertical; and show that, when this angle is chosen so as to make the acceleration as great as possible, the acceleration is

$$\frac{g}{(m_1+m_2+m_3)}\{m_1^2+m_2^2+m_3^2-m_2m_3-m_3m_1-m_1m_2\}^{\frac{1}{2}}.$$

27. A particle is shot along the rough upper surface of a board which is placed on a smooth inclined plane, the motion of both board and particle being parallel to the line of greatest slope. Find the accelerations of the particle and the board during their relative motion.

28. An isosceles wedge is placed with its base on a smooth horizontal plane, and a smooth particle is placed on each face and

allowed to slide down; if m, m' are the masses of the particles, M that of the wedge, find the inclination of either face of the wedge to the horizontal when the acceleration of the wedge is a maximum.

29. A bicycle track is made in the form of a circle sixty yards in diameter, banked up uniformly on the outside so that the radial inclination to the horizontal is $\cot^{-1} 3$. If the coefficient of friction be $\frac{1}{2}$, show that the wheels will not skid sideways unless the constant speed of the rider exceed 36·6 miles per hour. If the coefficient of friction be reduced to $\frac{1}{4}$, there will be an inferior limit to the constant speed; prove this, and find the limit.

30. Two sets of pulleys, both arranged according to the system in which there is only one string, are placed so that the same string passes round both sets, having both ends attached to the upper blocks, and passing from the highest pulley of the one set to the highest of the other set. Prove that, if there are n pulleys on each block with a weight W for one set and n' pulleys on each block with a weight W' for the other set, the acceleration of the highest part of the string will be

$$\frac{2nn'g(W'n - Wn')}{W'n^2 + Wn'^2}.$$

31. A weight M_1 is resting on a rough table, coefficient of friction μ; a string attached to it passes over a smooth pulley A at the edge of the table, under a second smooth pulley B vertically below, and then passes over a third C vertically above, after which it is fastened to a weight M_2 hanging freely.

If the pulleys A and C be fixed, but B be movable and of weight M, show that its acceleration is

$$g \cdot \frac{M(M_1 + M_2) - 2M_1 M_2 (1 + \mu)}{M(M_1 + M_2) + 4M_1 M_2},$$

provided that

$$\mu < \frac{3MM_2}{M_1(M + 4M_2)}.$$

32. In the first system of pulleys (in which each string is attached to a fixed beam) there are three movable weightless pulleys and one fixed pulley. The power and the weight are particles of equal mass. Prove that the acceleration of the power is $\frac{56}{65}g$, and that the tension of the string round the highest pulley is $\frac{9}{65}w$, where w is the weight of either of the particles.

33. In the system of pulleys in which all the strings are attached to the weight, show that if there be n pulleys of mass m, and if M and M' be the masses attached in the positions of weight and power, the acceleration of M downwards is

$$g \frac{M - m(2^n - n - 1) - M'(2^n - 1)}{M + m\left(\dfrac{2^{2n} - 4}{3} - 2^{n+1} + n + 3\right) + M'(2^n - 1)^2}.$$

34. $2n$ small smooth rings are fixed at equal intervals in a horizontal circle, and an endless string is passed through them in order. If the loops of the string between each alternate pair of rings support n pulleys of weight P, Q, R, ... etc., respectively, the portions of each loop not in contact with the pulleys being vertical, show that the pulley P will descend with acceleration

$$g\left\{\frac{\frac{1-n}{P}+\frac{1}{Q}+\frac{1}{R}+\dots}{\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\dots}\right\}.$$

35. Two particles of masses m and m' respectively, are attached to the ends of a string passing over a pulley A, and are held respectively on two inclined planes each of angle α, placed back to back with their highest edges vertically under the pulley. If each string make an angle β with the plane, show that the heavier particle m will at once pull the other off the plane if

$$\frac{m'}{m} < 2\tan\alpha\tan\beta - 1.$$

36. Two men A and B, each of mass m, sit in loops at the ends of a rope passing over a smooth pulley, A being h feet higher up than B. In B's hands is placed a ball $\left(\text{mass } \frac{m}{10}\right)$ which he instantly throws up to A so that it just reaches him. Show that by the time A has caught the ball he has moved up through a distance $\frac{2}{19}h$, and show also that he will cease ascending when he has ascended altogether a height $\frac{59}{361}h$.

37. Two small balls of equal masses, connected by an elastic string of natural length l, the tension of which is very great for even a small extension, are allowed to fall from rest from the same place, the motion of the second ball not being allowed to begin till the first ball starts it by means of the string. Describe the subsequent motion, supposing that the two balls can pass each other freely; and show that, after $(n+\frac{1}{2})\sqrt{\frac{2l}{g}}$ seconds, both balls will have fallen a distance $(n^2+\frac{1}{4})l$.

38. A parabola is placed in a vertical plane with its axis inclined at an angle α to the vertical and its vertex upwards. Prove that the time of quickest descent from the directrix to the curve is $\sqrt{\frac{l}{g}}\sec\frac{\alpha}{2}$, and that the time of quickest descent from the curve to the directrix is $\sqrt{\frac{l}{g}}\operatorname{cosec}\frac{\alpha}{2}$, where $2l$ is the latus rectum of the curve.

39. Construct the rough chord of quickest descent (angle of friction λ) to a given straight line inclined at an angle θ (which is greater than λ) to the horizon, from a point P above it in the same vertical plane; and if the distance of P from the straight line be h, prove that the time down the chord is

$$\sqrt{\frac{2h\cos\lambda}{g}}\,.\,\sec\frac{\theta+\lambda}{2}.$$

40. A particle of mass m is attached by two inelastic strings to particles of masses m', m'' respectively. The particles are placed at rest on a smooth horizontal table so that the two strings lie in two perpendicular straight lines. A blow is given to the particle m in the direction of the bisector of the angle between the strings, and in the sense such that the strings are jerked. Prove that the initial velocity of m' : initial velocity of m'' in the ratio $m+m'' : m+m'$.

41. Find the condition that a body resting on a smooth horizontal plane will tilt when acted on by an impulse.

A rectangular block of inelastic material is divided into two triangular prisms by a plane through two parallel and opposite edges. One of these is placed, hypotenuse upwards, on a smooth horizontal table, and the other held above it, hypotenuse downwards, with their corresponding triangular faces in the same planes and other corresponding faces parallel; the latter is then allowed to drop. Show that the former will not tilt unless $x < \frac{5}{6} - \frac{1}{2}\cos 2\alpha$, where x is the proportion of the length of one inclined face in contact with the other at the moment of impact, and α is their inclination to the horizon.

42. Two equal particles are connected by a string, one point of which is fixed, and the particles are describing circles of radii a and b about this point with the same angular velocity, so that the string is straight. The string is suddenly released; prove that the tensions in the two parts are altered in the ratios

$$(a+b) : 2a \quad\text{and}\quad (a+b) : 2b.$$

43. Two particles of masses m_1 and m_2 are attached by a fine inextensible string and move in a smooth horizontal plane with the same velocity v perpendicularly to the string. The string suddenly strikes against a smooth fixed post of small radius at distances l_1 and l_2 from m_1 and m_2. Prove that the initial acceleration of the point of the string initially touching the post is

$$v^2\left(\frac{m_1}{l_1} \sim \frac{m_2}{l_2}\right)\Big/(m_1+m_2).$$

44. One end of a string PQ is fixed to a point P in a smooth horizontal plane, and the other end Q is attached to a small smooth ring of mass m which rests on the plane; through the ring one end

of another string is passed and fastened to a point R of the plane, whilst a particle of mass M is attached to its other end S and rests on the plane. The angle PQR is obtuse and equal to β, the angle RQS is right. Show that, if the particle M be projected parallel to QR with velocity V, the initial tension in PQ will be

$$\frac{MmV^2}{a} \cdot \frac{\sin\beta - \cos\beta}{m + M + M\sin 2\beta},$$

where a is the length of QS.

45. A piece of string without weight, held in the form of a rhombus $ABCD$, has three equal heavy particles fastened to it at B, C, and D. The system is held so that AC is vertical by the application of horizontal forces at B and D. If the particles at B and D be let go simultaneously, prove that the tension of each of the portions AB, AD of the string is instantaneously changed in the ratio

$$1 + 2\cos^2\alpha : 3 + 6\sin^2\alpha,$$

where α is the inclination of each portion of the string to the vertical. Also find the initial acceleration of the particle at C.

46. Two masses M_1 and M_2 are connected by a spring. When M_1 is held fixed, M_2 vibrates once in T seconds; show that if M_2 is held, M_1 will vibrate once in $T\{M_1/M_2\}^{\frac{1}{2}}$ seconds; and find the time of vibration when both are free to vibrate.

47. A heavy particle is attached to one end of a fine elastic string, the other end of which is fixed. The unstretched length of the string is a, and its modulus of elasticity is n times the weight of the particle. The particle is pulled vertically downwards till the length of the string is a', and then let go from rest. Show that the time till it returns to this position is

$$2\left(\frac{a}{ng}\right)^{\frac{1}{2}}[\pi - \theta + \theta' + \tan\theta - \tan\theta'],$$

where θ, θ' are positive acute angles given by

$$\sec\theta = \frac{na'}{a} - n - 1, \quad \sec^2\theta' = \sec^2\theta - 4n;$$

and a' is so large that real values of the angles θ, θ' can be found.

48. Two particles of masses m and m' respectively are joined by an elastic string without mass, of modulus of elasticity λ and natural length a. Initially the particles are at rest on a smooth horizontal plane with the string at its natural length. One of the particles is projected directly away from the other. Prove that the string is next at its natural length after a time

$$\pi\sqrt{\frac{mm'a}{(m+m')\lambda}}.$$

49. An endless cord consists of two uniform portions of lengths $2l$ and $2l'$ and masses per unit length m, m' knotted together. If it be placed in stable equilibrium over a small smooth peg and slightly displaced, prove that its oscillations will be harmonic and of period

$$2\pi(ml+m'l')^{\frac{1}{2}}/(m \sim m')^{\frac{1}{2}}g^{\frac{1}{2}}.$$

50. A smooth pulley without weight is hung from a fixed point by an elastic string modulus λ, and unstretched length a; round this passes an inelastic string to the ends of which are attached heavy particles M, m; prove that the time of an oscillation of the pulley is $4\pi\left\{\dfrac{amM}{\lambda(M+m)}\right\}^{\frac{1}{2}}$.

51. $ABCD$ is a square, B and C being vertically above A and D respectively. A weightless elastic string of unstretched length equal to twice the side of the square, and for which the ratio of tension to extension per unit length is constant and equal to λ, passes round small smooth pulleys at B and C, has its ends fixed at A and D, and carries at its middle point a particle, the weight of which is 2λ; prove that in equilibrium the particle is at the middle point of AD, and that if it is placed initially at rest at the middle point of BC it will reach the position of equilibrium after a time

$$\frac{\pi}{2}\sqrt{\frac{AB}{g}}.$$

CHAPTER IV.

ON WORK AND ENERGY.

100. WHEN a force acts on a particle, the point which indicates the position of the particle is called the Point of Application of the force.

A straight line through the point of application in the direction of the force is called the Line of Action of the force.

101. When the point of application of a force is displaced, **work** is said to be done by or against the force.

Definition of the Work done by a Constant Force.

Let A be the original position of the point of application of a constant force P, AP the direction of the force, and let the point of application be displaced by any path to A'.

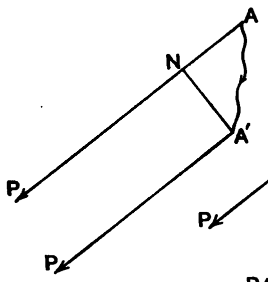


FIG. 1.

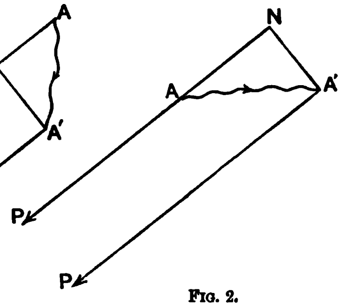


FIG. 2.

Draw $A'N$ perpendicular to AP or PA produced, meeting it in N. Then the work done by (or against) the force as its point of application is moved from A to A' is measured by $P \times AN$, *i.e.*

by *the product of the number of units in the force and the number of units in the resolved part of the displacement of the point of application parallel to the force*, the product being reckoned positive or negative according as the resolved displacement is (Fig. 1) in the same sense as, or (Fig. 2) in the opposite sense to, the sense of the force.

In the former case the work is said to be done *by* the force *on* the particle; in the latter case the work is said to be done *against* the force *by* the particle.

The definition shows that, when the point of application moves at right angles to the force, the force does no work.

The student should notice that the quantity of work done by a constant force as its point of application moves from A to A' is the same whatever path the point of application takes between A and A'.

102. Let θ be the angle (acute or obtuse) between the direction of a constant force P and the displacement AA' of its point of application.

Then in all cases the work done by P in the displacement $AA' = P \,.\, AN = P \,.\, AA' \cos\theta = (P\cos\theta) \,.\, AA'$ = work done by resolved part of P parallel to AA'.

Hence (see § 15) it follows that

When any number of constant forces act on a particle, the algebraic sum of the quantities of work done by the several forces while the particle undergoes any displacement is equal to the work done by their resultant.

DEFINITION. *This algebraic sum is called the work done by the system of forces.*

The measure of work is thus by definition a scalar quantity, and quantities of work (whether done by one or several forces) are compounded by algebraic addition.*

103. Unit of Work. The "absolute" unit of work is the work done by the absolute unit force (§ 63) when the resolved displacement of the point of application in the direction of the force is of unit length.

In the British system it is the work done by a poundal when the resolved displacement is of length 1 foot, and is called a *foot-poundal.*

In the C.G.S. system it is the work done by a dyne when the

* The product of two vectors is sometimes defined as a vector quantity, as in the case of a *moment* in three dimensions; we should have found that work, had we so defined it, would have had no physical significance.

resolved displacement is of length 1 centimetre, and is called an *erg*.

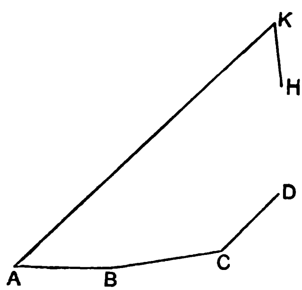
In England engineers employ a unit called the *foot-pound*, defined as the work done by a force equal to the weight of one pound when the resolved displacement of its point of application is equal to one foot.

The foot-pound thus varies with the value of g, and to obtain a strictly constant standard, the latitude at which the measurement is made should be defined.

The magnitude of the foot-poundal of course, like that of the poundal, is quite independent of locality. g foot-poundals make a foot-pound.

N.B.—Unless the contrary is expressly stated, the work in theoretical equations is measured in *absolute units*, usually either foot-poundals or ergs.

104. Work done by a Variable Force. It follows from the definition of work that, if a particle acted on by a constant force undergoes the successive displacements AB, BC, CD, ... HK, the whole work done in the displacement AK is equal to the algebraic sum of the quantities of work done in the separate displacements AB, BC, ... HK. This suggests the following definition of the work done by a variable force:



Let AB, BC, CD ... be successive displacements of the particle in successive short intervals of time, so that A, B, C, ... are points on the actual path.

While the particle undergoes any one of these displacements, *let the force be considered constant and equal to its value at the beginning of that displacement.*

Then, in general, as the successive displacements are indefinitely diminished, the algebraic sum of the quantities of work done by the constant forces, as the particle undergoes a finite displacement AK, *tends to a definite limit; this limit is the work done by the variable force during the displacement* AK.

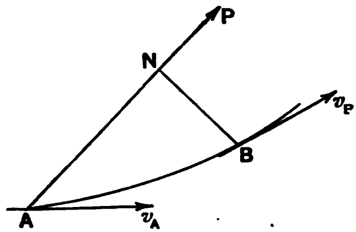
That this algebraical sum does in general tend to a definite limit will be shown in § 106.

The work done by a variable force during a given displacement AK *is not in general the same by all paths from* A *to* K.

We shall prove, however, that it *is* so in a large and important class of cases. (See § 115, below.)

105. The Equivalent of Work in terms of Mass and Speed.

Let a particle, mass m, move under the action of a constant force P from A to B, and let $d(=AN)$ be the magnitude of the



resolved displacement in the direction of the force. The acceleration of the particle is constant, and if f be its magnitude, $P=mf$.

Denoting by v_A, v_B the speeds at A and B respectively, we have from § 40, Chap. II., and with the same convention of signs as is there employed,

$$v_B^2 - v_A^2 = 2fd,$$

or, multiplying by $\frac{1}{2}m$,

$$\begin{aligned}\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 &= mfd \\ &= Pd \\ &= \text{work done by } P \text{ during the displacement } AB.\end{aligned}$$

Thus the quantity $\frac{1}{2}mv^2$ stands in an important relation to the work done on the particle; in fact the *change* in the value of this quantity in any displacement measures the work done during that displacement. It is convenient to have a distinguishing name for this quantity. Hence the following definition:

The product $\frac{1}{2}$mv², *where* m *is the mass of the particle,* v *its speed, is called the* **kinetic energy** *of the particle.*

Kinetic energy is thus an essentially positive scalar quantity.

The kinetic energy of a particle of unit mass moving with unit velocity is $\frac{1}{2} . 1 . 1$ units, *i.e.* half a unit. The unit of kinetic energy is that of a mass 2 moving with unit speed. There is no special name for this unit, as it is equal to the absolute unit of work.

106. We shall now prove the general proposition of which a particular case has just been given.

PROPOSITION. *If a force (constant or variable) acts on a particle, the increment of the kinetic energy during any displacement is equal to the work done by the force during that displacement.*

We shall assume as usual that there is a velocity and a finite acceleration at every point of the particle's path.

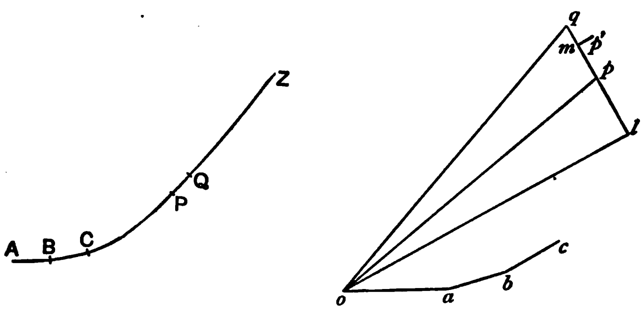
Let $A, B, C, \dots P, Q, \dots Z$ be n points on the path, the displacements $AB, BC, \dots PQ, \dots$ being each made in a small interval $\frac{t}{n}$, where t is the time from A to Z; and let $f_a, f_b, \dots f_p, \dots$ be the accelerations *at* $A, B, \dots P, \dots$ respectively.

Let oa represent the velocity at A, and draw

$$\overline{ab} = \bar{f}_a \cdot \frac{t}{n}, \ \overline{bc} = \bar{f}_b \cdot \frac{t}{n}, \ \dots \ \overline{pq} = \bar{f}_p \cdot \frac{t}{n}, \dots \text{ etc.}$$

Then (§ 26, Chap. II.), when n is increased indefinitely, the displacement of q from the point on the hodograph corresponding to Q diminishes indefinitely.

Let op' represent the *average velocity* from P to Q.*



Then as n increases the sides of the triangle pqp' all diminish indefinitely. (Def., § 24.)

Draw $ol, p'm$ perpendiculars on pq, produced if necessary.

Then we have the scalar relation

$$\begin{aligned} oq^2 - op^2 = lq^2 - lp^2 &= \pm(lp + lq) \cdot pq \\ &= \pm 2lm \cdot pq + 2e_p \cdot pq, \end{aligned}$$

where e_p is

* To avoid complicating the figure, o and p' are not joined.

a quantity, positive or negative, numerically less than the longest side of the triangle pqp',

$$= \pm 2lm \,.\, f_p \,.\, \frac{t}{n} + 2e_p \,.\, f_p \,.\, \frac{t}{n},$$

since $$pq = f_p \,.\, \frac{t}{n}.$$

Further, $lm \,.\, \frac{t}{n} =$ (resolved part of average velocity in direction of acceleration at $P) \times \frac{t}{n}$

= resolved displacement in this direction

$= d_p$, say.

Thus, adopting the usual convention (§ 40, iv.) for the signs of d_p and f_p, we have

$$oq^2 - op^2 = +2d_p \,.\, f_p + 2e_p \,.\, f_p \,.\, \frac{t}{n}.$$

Writing down a corresponding equation for each pair of points a, b; b, c; c, d; ... etc, and adding the equations, we have

$$oz^2 - oa^2 = 2\Sigma f_p \,.\, d_p + 2\Sigma e_p \,.\, f_p \,.\, \frac{t}{n},$$

where z, not shewn in the figure, is the last of the series of points $a, b, c \ldots$.

Now if E be the numerical value of the numerically greatest of the quantities $e_a, e_b, \ldots$, and if F be the numerical value of the numerically greatest of the quantities $f_a, f_b, f_c, \ldots$, the last term on the right-hand side

$$< 2n \,.\, E \,.\, F \,.\, \frac{t}{n}, \quad < 2E \,.\, F \,.\, t,$$

and therefore diminishes indefinitely when n increases, even when all the quantities $e, f, \ldots$ are positive.

Increasing n indefinitely, $\overline{oz}$ becomes the velocity at Z.

Hence, denoting the speeds at A, Z by v_A, v_Z respectively,

$$\tfrac{1}{2}(v^2{}_Z - v^2{}_A) = \text{the limit of } \Sigma fd,$$

or $$\tfrac{1}{2}m(v^2{}_Z - v^2{}_A) = \text{the limit of } \Sigma mfd.$$

The term on the left is the change of the kinetic energy during the displacement AZ. That on the right is the work done on the particle during this displacement, as defined in § 104.

A similar theorem would evidently be true if we took the forces to have their values at the ends or any intermediate points of the intervals.

We have proved incidentally that the sum whose limit, if any, we have defined as the work, *has* in general a definite limit.

The restriction with which we started shows that no *impulse* must act on the particle during the displacement.

In the above investigation, the motion of the particle and the line of action of the force need not be confined to one plane.

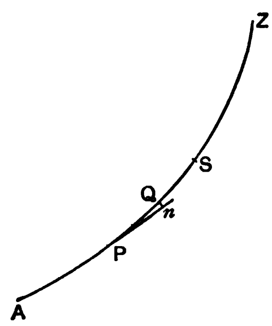
107. It is clear from the preceding paragraph that the kinetic energy of a particle at a given instant measures the work that could be done by the particle against any system of forces before being brought to rest. This is the origin of the name "kinetic energy," which means the energy or capacity for doing work that the particle possesses in virtue of its motion.

108. Work done by Tangential and Normal Components. When the particle is at a point P of its path, let the force acting on it be resolved into two components T and N parallel respectively to the tangent and normal *at* P, and let similar resolutions be made at the beginning of each small displacement.

Then since the forces are to remain *constant* during each of the small displacements,

The work done on the particle during the displacement AZ = work done by tangential components + work done by normal components (§ 102).

Let now the tangential component at each point of the path be *zero*; the *speed-acceleration* is therefore zero (§ 49), and therefore the *speed is constant*, so that the kinetic energy is constant, and the forces do no work. Hence the *normal components* are in this case to be regarded as doing no work; and as the work done by them is clearly independent of that done by the tangential components, *the normal forces are in every case to be regarded as doing no work.*



The reader acquainted with the elements of the Differential Calculus will see that the work done by the normal component N during the displacement PQ is a small quantity of the second order, PQ being of the first.

If Pn be the tangent at P to the path, and Qn be a perpendicular from Q on this tangent, the work done in any displacement may thus be regarded as the limit of the quantity $\Sigma T.Pn$. This has

the same limit as $\Sigma T(arc\ PQ)$, as may be shown thus: Divide the path AZ into finite portions such that the speed acceleration has the same sign throughout each: let AS be the first of these in which the speed acceleration is (say) positive. Then by a well-known theorem in algebra, the ratio

$$\frac{\Sigma_A^S T(arc\ PQ)}{\Sigma_A^S T.\,Pn}$$

lies between the greatest and the least of the series of fractions

$$\frac{T(arc\ PQ)}{T.\,Pn},$$

that is, of the series of fractions

$$\frac{arc\ PQ}{Pn};$$

and each of these fractions has *unity* for its limit, since there is supposed to be a velocity at each point of the path. Similar reasoning applies for each successive portion of the path for which the speed acceleration keeps one sign.

Corollary. If v_P, v_Q be the speeds at P and Q, two neigbouring points of the path, we have

$$\text{limit of } \frac{\frac{1}{2}mv_Q^2 - \frac{1}{2}mv_P^2}{arc\ PQ} = T,$$

the resolved force parallel to the tangent at P, or

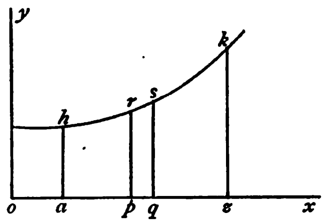
The space rate of change of kinetic energy at any point of a particle's path is equal to the resolved force along the tangent at that point.

The student acquainted with the elements of the Differential and Integral Calculus will see that the work done on a particle between A and Z may be written as the integral $\int_A^Z Tds$, and that the last result may be written

$$\frac{d}{ds}\left(\frac{1}{2}mv^2\right) = T.$$

109. The Curve of Work. The relation between the kinetic energy of a particle and the work done on it may be represented graphically as follows: Taking axes ox, oy, let the abscissa op represent the length of the path measured from a fixed point up to a point P on the path, and let the ordinate pr represent the resolved force along the tangent at P. Then the rectangle whose adjacent sides are rp, pq represents the term $T(arc\ PQ)$. As the particle moves from A to Z, p will move

from a to z, and r will trace out a curve hk. By the principle of § 7, the area included between this curve, the axis of x, and the extreme ordinates represents the limit of the sum $\Sigma T(arc\,PQ)$, and therefore also the change of kinetic energy as the particle moves from A to Z. The curve hk is called the curve of work.



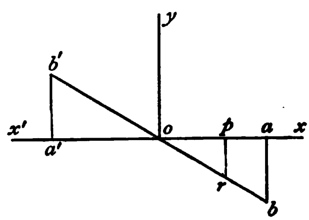
If the particle turns and retraces its path, the point p may be supposed to move in the negative sense along the axis of x, but this is not necessary; it is sometimes convenient to suppose that p moves continuously in the positive sense; op then represents the actual length of path traversed.

When the speed and speed-acceleration are in the same sense, the work being done on the particle at the instant is positive; when they are in opposite senses, it is negative. The axes being drawn as above, it follows that *in whichever sense the representative point* p *is moving, areas are to be reckoned positive if to the left hand, negative if to the right hand, of a person looking along the axis of x in this sense.*

The curve of work may also be drawn by taking the successive increments pq of abscissa equal to the resolved parts of the small displacements PQ in the directions of the *resultant* forces, and the ordinates to represent the magnitudes of the resultant forces, at the beginning of each small displacement. This method is especially convenient when the force acting on the particle is directed to a fixed point.

Examples.

1. Find, by means of the curve of work, the velocity in simple harmonic motion.



If o represents the position of equilibrium, aa' the amplitude, the curve of work is the straight line bob', where $\tan boa$ is numerically equal to $m\mu$, the force at unit distance, m being the mass of the particle, and μ the value of the acceleration at unit distance.

The work done by the force is positive while p moves inwards to

o, negative while p moves outwards from o. When p is at a' the kinetic energy is zero. In the position denoted by p it is

$$\begin{aligned}\tfrac{1}{2}mv_p^2 &= \text{area } oa'b' - \text{area } opr\\ &= \tfrac{1}{2}oa' \,.\, a'b' - \tfrac{1}{2}op \,.\, pr\\ &= \tfrac{1}{2}m\mu(oa'^2 - op^2),\end{aligned}$$

or putting $$oa = oa' = c, \quad op = x,$$

$$v_p^2 = \mu(c^2 - x^2), \text{ the result of § 50.}$$

2. Prove that the work done in stretching in a straight line an elastic string which obeys Hooke's Law is equal to the product of the extension and the arithmetic mean of the initial and final tensions, and that this formula holds even if the string becomes slack during some part of the motion.

3. A particle of mass m is acted on by a force whose direction is constant, and which, as the particle describes a straight line, varies uniformly with the space described from zero to a given value P, which it reaches after the particle has described a space a; during the next space $2a$, the force varies uniformly from P to $-P$, then during the next space $2a$ from $-P$ to $+P$, and so on. Draw the curve of work and determine the kinetic energy of the particle in any position, supposing it to start from rest.

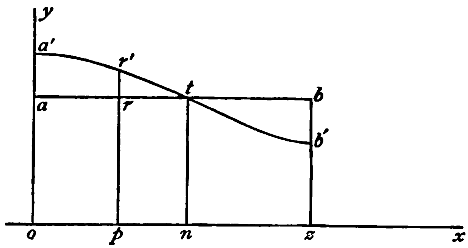
4. A rod, the tension in which obeys Hooke's Law, is in a vertical position with its upper end fixed. A weight is suddenly attached to the lower end. Neglecting the mass of the rod, find the greatest tension and extension produced by the weight.

A weight suddenly attached in this way is called by engineers a "live load." Prove that for such a rod the engineers' rule which estimates "live loads" as equivalent to "dead loads" of twice the amount is strictly accurate.

5. A particle lies on a rough horizontal table, and is attached to an elastic string which passes through a small hole in the table; the other end of the string is fixed vertically below the hole at a distance equal to its natural length. The particle is placed at a distance away from the hole, released, and oscillates. Draw the curve of work for the whole motion.

6. A child's ball of mass m is attached to the hand by an elastic string of natural length l and modulus λ. It is dropped vertically downwards from the hand, which is held in the same position till it feels the string slackened on the upward rise of the ball. How much must the hand be lowered to catch the ball at the summit of its upward rise, and how long after the release of the ball will this occur?

110. The work done by a man in raising a mass m to a height h is numerically equal to mgh, provided that the mass is initially and finally at rest. By means of the curve of work we can discuss the way in which this work is in practice distributed.



Let oz represent the height h, and when the mass is at a height represented by op, draw pr' to represent the force exerted by the man at this instant. As p moves from o to z, r' will trace out the curve $a'r'tb'$ whose area represents the work done by the man.

Take oa to represent the weight mg, and complete the rectangle $oabz$, which will thus represent the work done by the mass against gravity.

Now, at first, in order to set the mass in motion, the man will exert a force somewhat greater than mg, and the curve will lie above ab. Also, at the end of the process, in order that the mass may come to rest at height h, the force exerted by the man will be less than the weight, and the curve will lie below ab.

The area $oa'tb'$ is equal to the area $oabz$, each being equal to mgh.

The area $aa'r'r$ represents the kinetic energy of the mass when at a height op.

If t be the point where $a'b'$ cuts ab, the mass is losing kinetic energy after r' passes t. If tn be the ordinate of t, on represents the height at which the speed is a maximum.

If the man raises the weight *very slowly*, the curve $a'b'$ may be made to approach as nearly as we please to the straight line ab.*

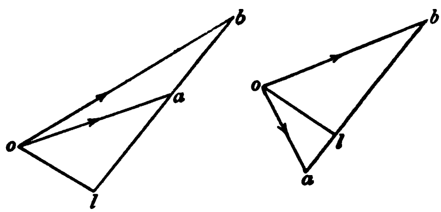
In the application of work to statical problems, the displacement being finite and supposed made "infinitely slowly," some such consideration as the above is implied.

* The idea of this paragraph is borrowed from *Tracts on Mechanics* by M. W. Crofton, F.R.S.

111. Work done by an Impulse. A theorem similar to that given in § 105 for forces applies to impulses. An "instantaneous" impulse does not alter the position of the particle to which it is applied; it would at first sight, then, appear to be absurd to speak of the work done by an impulse; an impulse, however, changes the kinetic energy of the particle, and we may regard this change of kinetic energy as an equivalent of work done (*on or by the particle* according as the kinetic energy is increased or decreased). Then, further to justify our calling this work done *by or against the impulse,* we note that an impulse is equivalent in its effect (as measured by the momentum produced) to a very great force acting for a very short interval of time. In this very short interval the particle will have *some very small displacement,* and the product of the large force and the resolved part of this small displacement in the direction of the force will in general be finite, however small the interval of time be taken to be, that is, however nearly the circumstances approximate to those of an *instantaneous impulse.*

Proposition. *When an instantaneous impulse acts on a particle in any direction, the work done is equal to the product of the impulse and half the sum of the speeds of the initial and final velocities resolved in the direction of the impulse.*

Or, in symbols, if I be the measure of the impulse, u and v the speeds of the resolved parts of the initial and final velocities in the direction of the impulse, the work done $=I\,.\,\dfrac{u+v}{2}$.



The work done by the impulse is measured by the kinetic energy gained. Let m be the mass of the particle, and let oa represent the initial, ob the final velocity.

The change of velocity is represented by ab, and the impulse I is measured by $m\,.\,ab$, ab being its direction.

Draw ol perpendicular to ab or ab produced.

Then the vectors la, lb represent the resolved velocities in the

direction of the impulse; the lengths of la, lb represent the speeds.

Then the gain of kinetic energy $=\frac{1}{2}m(ob^2-oa^2)$

$$=\tfrac{1}{2}m(lb+la)\,.\,ab$$

$$=\tfrac{1}{2}m\,.\,ab(lb+la)=I\,.\,\frac{u+v}{2}$$

$=I\times$(half the algebraic sum of the speeds of the initial and final velocities resolved in the direction of I).

There are other varieties of the figure, which should be drawn by the student.

Examples.

1. What is the graphical condition that kinetic energy should be gained by the impulse?

2. Show that when a particle is moving with a given kinetic energy it is always possible so to apply a given impulse that its kinetic energy may remain unaltered, provided this impulse is less than that which would generate in the particle at rest a kinetic energy equal to four times that which it actually possesses.

3. A mutual attractive impulse acts between two particles of masses m and m' which are separating with relative velocity V; prove that kinetic energy is gained or lost according as the impulse is greater or less than $\dfrac{2mm'}{m+m'}\,.\,V$.

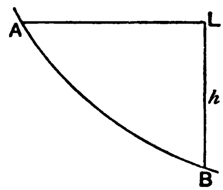
4. Show that the energy lost when a particle which is moving freely is suddenly constrained to alter its motion by the tightening of a string fastened to it and to a fixed point is equal to the energy due to its motion before the impulse in the direction assumed by the string when tight.

5. Two weights P and Q $(P>Q)$ are attached to the ends of a string which hangs over a smooth pulley. A third weight R $(R+Q>P)$ rests on an inelastic plane vertically below Q, attached to Q by a string. If P is allowed to descend and the system left to itself, prove that it will come to rest after a time

$$\frac{4V}{g}\,\frac{P(P+Q)}{(P-Q)(R+Q-P)}$$

from the instant at which the string connecting Q and R first becomes taut, where V is the common speed of P and Q just before the first impulsive action.

112. *A particle starts with given speed to slide down a smooth fixed curve under gravity; to find its speed as it passes any other point of the curve, and the reaction of the curve.*



Let m be the mass of the particle, u its speed as it starts from a point A, v its speed at another point B, whose *vertical* distance below A is h.

Since the curve is smooth, the reaction is always along the normal through the particle, and therefore (§ 108) does no work.

$$\therefore \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$
$$= \text{work done by the weight in the displacement } AB$$
$$= mgh,$$

or $$v^2 = u^2 + 2gh,$$

giving the same value for v as if the particle had fallen freely from a height h with initial vertical velocity u.

Next let N be the value of the reaction at B, R the radius of curvature of the curve at B, ψ the inclination of the tangent at B to the horizon. The acceleration inwards along the normal is $\frac{v^2}{R}$ (§ 49).

Hence, resolving in this direction,

$$m \cdot \frac{v^2}{R} = N - mg\cos\psi,$$

or $$N = m\left(\frac{u^2+2gh}{R} + g\cos\psi\right).$$

Given the intrinsic equation to a smooth curve, the curve of work for a particle sliding down it under gravity can at once be written down. In fact if $s = F(\psi)$ be the intrinsic equation, taking $m = 1$ and writing $x = s$, $y = g\sin\psi$, we have $x = F\left[\sin^{-1}\frac{y}{g}\right]$ for the curve of work; conversely, from the curve of work in this case we can write down the intrinsic equation to the path.

Examples.

1. A bead of mass m slides on a vertical smooth circular wire, radius a, starting from rest at the highest point. Find the speed at any point and the pressure on the wire.

[When the radius through the bead makes an angle θ with the upward vertical, the speed is $2\sin\frac{\theta}{2}\sqrt{ga}$. The pressure is then $mg(3\cos\theta-2)$ absolute units.]

2. If the particle slides down from rest at the highest point on the outside of a smooth sphere, where will it quit the surface?

[This happens when the pressure changes sign, or at a vertical distance equal to $\frac{1}{3}$ of the radius below the highest point.]

3. A particle is attached to a light string of length l which hangs vertically. What horizontal blow must be given to the particle in order that it may just perform complete revolutions in a vertical plane?

4. Prove that when the particle is performing complete revolutions, the sum of the tensions, when it is at the opposite ends of any diameter, is the same for all diameters.

5. A rough wire, whose curvature is continuous and in the same direction but very small, is placed in a vertical plane with its concavity upwards. A bead is placed at rest on the wire, and after sliding down it for a short distance again comes to rest. Give an approximate geometrical construction for the point at which this happens.

6. A heavy ring slides along a smooth vertical circular wire, being projected from any point. Show how to represent graphically the periodic nature of the changes in the kinetic energy of the ring, discussing the cases (1) when the ring performs complete revolutions, (2) when the motion is oscillatory.

[If the length of path traversed be taken for abscissa, the curve of work is the sine curve.]

7. A particle moves down a smooth curve under gravity. If the curve of work {abscissa = length of path} is a straight line, the smooth curve must be a cycloid.

[The intrinsic equation to the cycloid is $s=c\sin\psi$.]

8. A smooth elliptic wire is fixed with its major axis vertical. A smooth bead of weight W is placed on it at its highest point, and allowed to slide down. Show that the pressure on the wire at the extremities of the minor axis and the lowest point are $2nW$ and $\left(1+\frac{4}{n^2}\right)W$, where n is the ratio of the minor to the major axis.

9. When a bead performs complete revolutions on a smooth vertical circular wire, the velocity being that due to a fall from a horizontal

line HK, prove that, I being the internal limiting point of the coaxial system of circles of which HK is the radical axis and the given circle is one, then any chord through I divides the wire into two parts the times of describing which are equal.

113. We will now discuss the theory of **work as applied to a system of particles.** We thus first extend our definition of work:

DEFINITION. *The work done by the forces acting on a system of particles during any change of configuration of the system is the algebraic sum of the quantities of work done by the several forces.*

As an example, we may take the following proposition:

The work done in raising a number of masses $m_1, m_2, m_3 \ldots$ *against gravity through vertical heights* $y_1, y_2, y_3, \ldots$ *is equal to the work done in raising a mass equal to* $m_1+m_2+m_3 \ldots$ *through a height equal to the vertical displacement of their centre of mass.*

For the work done (in absolute units)

$= m_1gy_1 + m_2gy_2 + m_3gy_3 + \ldots$ by definition

$= g(m_1y_1 + m_2y_2 + m_3y_3 + \ldots)$

$= gy\Sigma m$, where y is the vertical displacement of the centre of mass

$=$ work done in raising a mass Σm through a vertical height y.

DEFINITION. *The kinetic energy of a system is the sum of the kinetic energies of its particles.*

That is, if $m_1, m_2, m_3, \ldots$ be the particles, $v_1, v_2, v_3, \ldots$ their speeds, the kinetic energy of the system

$$= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \ldots = \tfrac{1}{2}\Sigma mv^2.$$

It is, of course, an essentially positive quantity.

From the definition of work just given and from § 106 it follows that the excess (or defect) of the value of the quantity $\frac{1}{2}\Sigma mv^2$ in a configuration B of the system over (or from) its value in a configuration A measures the work done by (or against) the forces acting on the system as the system moves from the configuration A to the configuration B.

If the kinetic energy of the system (and therefore the speed of each particle) in the configuration B is zero, the kinetic energy in the configuration A is evidently equal to the work done against the forces while the system moves from the configuration A to the configuration B.

The kinetic energy of a system thus measures the work the system can do in virtue of its motion.

Example. The mass of a flywheel is 10,000 lbs., and it is of such a size that its mass may all be supposed concentrated on the circumference of a circle 12 ft. in radius. What is its kinetic energy when making 15 revolutions a minute?

114. The kinetic energy of a system calculated relative to any set of kinetic axes may be conveniently broken up into two parts, one of which depends on the motion of the particles of the system relative to the centre of mass, the other on the motion of the centre of mass relative to the kinetic origin.

Let u, v, w be the resolved parts of the velocity of any particle m relative to the kinetic origin and parallel to the kinetic axes, U, V, W the resolved parts of the velocity of G the centre of mass of the system in the same directions, u', v', w' the resolved parts of the velocity of m relative to G. Then

$$u = U + u', \quad v = V + v', \quad w = W + w';$$

and the kinetic energy of the system

$$\begin{aligned} &= \tfrac{1}{2}\Sigma m(u^2 + v^2 + w^2) \\ &= \tfrac{1}{2}\Sigma m[(U+u')^2 + (V+v')^2 + (W+w')^2] \\ &= \tfrac{1}{2}(U^2 + V^2 + W^2)\Sigma m + \tfrac{1}{2}\Sigma m(u'^2 + v'^2 + w'^2), \end{aligned}$$

since the terms $\Sigma mUu'$, $\Sigma mVv'$, $\Sigma mWw'$ are respectively equal to $U\Sigma mu'$, $V\Sigma mv'$, $W\Sigma mw'$, and $\Sigma mu' = \Sigma mv' = \Sigma mw' = 0$ by a property of the centre of mass.

The term $\frac{1}{2}\Sigma m(u'^2 + v'^2 + w'^2)$ is conveniently called the kinetic energy due to the motion of the system relative to the centre of mass, or as an abbreviation the *kinetic energy relative to the centre of mass.*

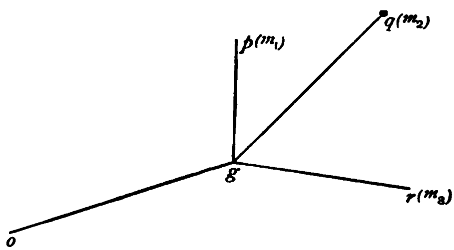
The term $\frac{1}{2}(U^2 + V^2 + W^2)\Sigma m$ is the kinetic energy of a particle whose mass is equal to the sum of the masses of the particles of the system, moving with the velocity of the centre of mass. It is conveniently called the *kinetic energy of the centre of mass.* Hence as a mnemonic we have the formula,

Kinetic energy of a system = kinetic energy relative to the centre of mass + kinetic energy of the centre of mass.

The latter term cannot be altered by the *internal* forces of the system, since such forces leave the velocity of the centre of mass unaltered.

Corollary 1. The kinetic energy of a system relative to its centre of mass is less than its kinetic energy referred to any kinetic origin other than the centre of mass. This follows since $\frac{1}{2}(U^2+V^2+W^2)\Sigma m$ is a positive quantity.

Corollary 2. If one or more of the masses of the system *explodes* (*i.e.* is forced by internal stress into several parts), the kinetic energy of the system is increased; and if several particles *solidify* into one, the kinetic energy of the system is *diminished* (Carnot's Theorem).



For let the mass m, moving with velocity og, explode into the fragments m_1, m_2, m_3, ... moving with velocities op, oq, or, ...

The velocity of the centre of mass of m_1, m_2, m_3 ... is, since the stresses are *internal*, unaltered by the explosion (§ 86). It is, therefore, still og after the explosion.

Hence the kinetic energy of the mass m which was $\frac{1}{2}m\,.\,og^2$ becomes

$$\tfrac{1}{2}(m_1+m_2+m_3+\dots)og^2+\tfrac{1}{2}m_1\,.\,gp^2+\tfrac{1}{2}m_2gq^2+\dots,$$

or $\frac{1}{2}m\,.\,og^2+$positive terms.

The kinetic energy is therefore *increased* by the explosion.

Similarly, if several particles m_1, m_2, m_3, ... coalesce into a particle of mass $m_1+m_2+m_3+\dots$, the kinetic energy of the system will be *diminished* by the suppression of the positive terms $\frac{1}{2}m_1gp^2+\frac{1}{2}m_2gq^2+\dots$.

So also, if two particles of the system are connected by an inextensible string which suddenly becomes tight, the kinetic energy of the system is diminished by the suppression of the terms due to the resolved parts in the direction of the string of the velocities of the two particles relative to their centre of mass.

Examples.

1. If three particles m_1, m_2, m_3 moving with velocities v_1, v_2, v_3 making angles θ_{23}, θ_{31}, θ_{12} with each other impinge and coalesce, the loss of energy is

$$\frac{\Sigma m_1 v_1^2 (m_2+m_3) - 2\Sigma m_2 m_3 v_2 v_3 \cos\theta_{23}}{2(m_1+m_2+m_3)}.$$

2. Two particles united by a fine inextensible string of length l are set in motion on a smooth horizontal plane with a total kinetic energy E. Prove that the tension of the string is greatest when the motion is such that the centre of gravity is at rest, and that it is then equal to $\frac{2E}{l}$.

115. Conservative Systems.

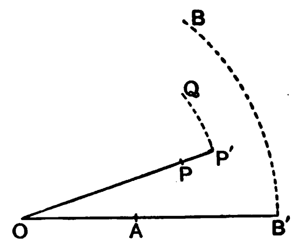
Definition. *When the work done by the several forces on the particles of a system as the system passes from one configuration to another is the same for all paths which the particles can take, the system is said to be Conservative, and the forces are called Conservative Forces.*

It is characteristic of conservative forces that their magnitudes and directions do not depend on the velocities of the particles to which they are applied. Also in any given position of a particle, the force applied to it can have one value only.

The most important classes of conservative forces are :

(i.) *Constant Forces.* That these are conservative is manifest from the definition of work (§ 101).

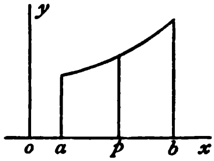
(ii.) *Forces directed to fixed centres and whose magnitude depends only on the distances from those centres of the particles on which they act.*



Let such a force, directed to a fixed point O, act on a particle which is displaced by any path from A to B. Then during a small displacement PQ the force is to be regarded as *constant* (Definition, § 104). The work done from P to Q is therefore the same as if the particle proceeded in the straight line OP to a point P' such that $OP' = OQ$, and then along a circle centre O from P' to Q. The work done in the latter displacement is zero (§ 108).

Hence the work done in the displacement PQ is equal to

force at $P \times PP'$. The force at P depends only on the *length* OP, not on its direction. Hence, taking the abscissa op of the curve of work equal to the length OP ($=r$, say) and the ordinate equal to the force at a distance r from O, we see that the curve of work is the same for all paths from A to B.



For example, the work done by an elastic string attached to O and to a particle which moves from A to B = work done as the particle moves in the straight line OAB' till $OB' = OB$, and then in a circle centre O from B' to B. The work done as the particle moves in the circle is zero (§ 108). Therefore the total work done

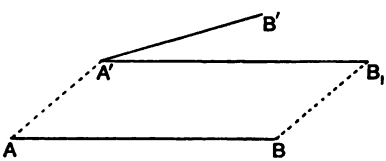
$$= \frac{\text{tension at } A + \text{tension at } B}{2}(OB' - OA) \quad (\S 109, \text{Ex. } 2)$$

$$= \frac{\text{tension at } A + \text{tension at } B}{2}(OB - OA).$$

Example. A light elastic string has its ends fastened to two points in a smooth horizontal plane; on the string slides a small heavy ring. Show that, if the ring be projected with a given speed in any direction and from any position where the string is unstretched, it will always reach but never get beyond a certain ellipse.

(iii.) *Newtonian stress-pairs of reactions between particles*, provided that the magnitude of the reaction between any pair of particles A, B depends only on the distance AB.

Let AB, $A'B'$ be two neighbouring positions of the line joining the particles, R the reaction between them when their positions



are A, B. The reaction is to be regarded as constant while A goes to A' and B to B' (§ 104). Hence we may make the displacement by moving AB parallel to itself to $A'B_1$, and then moving the particle at B_1 to B'. In the former displacement equal and opposite amounts of work are done on the particles; in the latter the work done $= R$ (extension of AB), and the result follows exactly as in case (ii.).

In particular, the reactions between two particles rigidly connected, and therefore all the reactions between the pairs of particles of a rigid body, do no work in any displacement.

As an example, it follows from the proposition given in § 113 that the work done against gravity on a rigid body of mass M when its centre of mass is raised through a vertical height y is Mgy, however the body be turned round its centre of mass during the operation.

It is also clear that the increase of the kinetic energy of a rigid body is equal to the work done by the *external* forces acting on the body.

Example. Prove from the theory of work that, if two particles connected by a light string be projected in any manner under gravity so that the string remains tight, the angular velocity of the string is constant.

116. Potential Energy of a Conservative System. Conservation of Energy.

DEFINITION. *The Potential Energy of a conservative system in any configuration* A *is the work that can be done by the forces on the particles, as the system passes from the configuration* A *to a standard configuration* A_0.

The standard configuration A_0 is usually so chosen that the potential energy in all other configurations considered is positive.

The potential energy in the configuration A depends solely on the positions of the particles in that configuration and not at all on their velocities or the paths by which they arrive.

If the system is started in the configuration A with any given kinetic energy, it will not of necessity pass through the configuration A_0. If, however, B denote any configuration through which it does pass, and if V_A and V_B denote the values of the potential energy, T_A and T_B those of the kinetic energy, in the positions A and B respectively, it is evident, since the work done is independent of the path, that $V_A - V_B =$ work done by the forces as the system passes from the configuration A to the configuration B.

But (§ 113) this quantity of work also equals the increment $T_B - T_A$ of the kinetic energy;

$$\therefore \quad V_A - V_B = T_B - T_A,$$

or

$$T_B + V_B = T_A + V_A,$$

that is,

The sum of the kinetic and potential energies of a conservative system is the same in all configurations.

This is the principle of the conservation of energy as used in Mechanics. For a more complete statement, see §§ 121-123.

As a simple example, consider a mass m projected vertically upwards from the surface of the Earth with speed u, and let v be its speed at any height h. Taking the standard position of potential energy at the point from which the mass starts,

$$\text{kinetic energy} + \text{potential energy} = \tfrac{1}{2}mv^2 + mgh$$
$$= \tfrac{1}{2}mu^2 + 0,$$

which gives $v^2 = u^2 - 2gh$, the equation obtained in § 43.

117. It will be convenient to summarize here the class of forces which do no work, and which therefore do not enter into the equation of energy. These are

(i.) The reactions between the particles of a rigid body (§ 115 (iii.)).

(ii.) The tension of a light inextensible string. Equal and opposite amounts of work are done by the tension on particles attached to either end. In particular, if one end is fixed, the tension at the other will do no work.

(iii.) The reaction of a smooth fixed surface. [If the surface is moving, the reaction may do work, though the stress-pair, of which it forms a part, will not. The student will find a case in point in Ex. 17 at the end of the chapter.]

(iv.) The reaction and friction of one surface rolling without slipping on another fixed surface. That the latter, called *Rolling Friction*, does no work may be gathered from § 52; the rolling surface is to be regarded for a small interval as turning round its point of contact. [If both surfaces are moving, still the stress-pair of reactions and the stress-pair of frictions will do no work.]

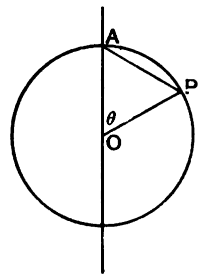
These forces are sometimes called "geometrical forces" or constraints, their effect being merely to impose certain kinematical conditions on the moving body.

Examples.

1. A ring P slides on a circular wire placed with its plane vertical. It is connected with the highest point A of the wire by means of an elastic string whose natural length is equal to the radius of the

circle, and starts from rest in the position in which the string is inclined to the vertical at an angle $\frac{\pi}{3}$. Show that, if it be again reduced to rest at the lowest point, the modulus of elasticity is three times the weight of the ring. Prove also that the pressure of the ring on the wire will be least when the string is inclined to the vertical at an angle $\cos^{-1}\left(\frac{5}{8}\right)$.

Let m be the mass of the ring, a the radius of the wire, θ the inclination of the radius through the ring to the upward vertical when the speed of the ring is v. The length of the string is then $2a\sin\frac{\theta}{2}$; hence the tension is



$$\lambda \cdot \frac{2a\sin\frac{\theta}{2}-a}{a}, \text{ or } \lambda\left(2\sin\frac{\theta}{2}-1\right),$$

and the work the string could do before becoming slack is

$$\frac{\lambda a}{2}\left(2\sin\frac{\theta}{2}-1\right)^2.$$

The only forces doing work on the ring are the tension of the string and the weight. Hence the potential energy in the position defined by θ may be written

$$C+\frac{\lambda a}{2}\left(2\sin\frac{\theta}{2}-1\right)^2+mga\cos\theta,$$

the constant C being added when it is not necessary to define the standard configuration.

The energy equation is therefore

$$\frac{1}{2}mv^2+\frac{\lambda a}{2}\left(2\sin\frac{\theta}{2}-1\right)^2+mga\cos\theta=\text{constant}. \quad \text{..........(1)}$$

In this equation write (i.) $v=0,\ \theta=\frac{\pi}{3}$;

(ii.) $v=0,\ \theta=\pi$,

and from the resulting identity we at once have $\lambda=3mg$.

Give λ this value; then resolving along the inward normal to the wire, we have, if R be the reaction,

$$\frac{mv^2}{a}=3mg\left(2\sin\frac{\theta}{2}-1\right)\sin\frac{\theta}{2}+mg\cos\theta-R, \quad \text{..............(2)}$$

whence from (1) and (2),

$$3mg\left[\left(2\sin\frac{\theta}{2}-1\right)^2+\left(2\sin\frac{\theta}{2}-1\right)\sin\frac{\theta}{2}\right]+3mg\cos\theta=\text{constant}+R.$$

R is therefore a maximum or minimum when

$$\left(2\sin\frac{\theta}{2}-1\right)\left(3\sin\frac{\theta}{2}-1\right)+\cos\theta$$

is a maximum or minimum, *i.e.* when

$$4\sin^2\frac{\theta}{2}-5\sin\frac{\theta}{2}+2$$

is a maximum or minimum.

The methods of Algebra or the Differential Calculus give at once

$$\cos\frac{\theta}{2}=0, \text{ or } \sin\frac{\theta}{2}=\frac{5}{8}.$$

The former gives $\theta=\pi$; in the latter the string is inclined to the vertical at an angle

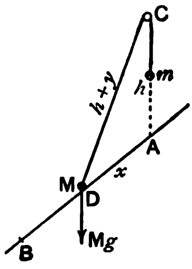
$$\frac{\pi}{2}-\frac{\theta}{2}, \text{ or } \cos^{-1}\frac{5}{8}.$$

The reaction on the ring, being outwards, is also a maximum when the ring starts $\left(\theta=\frac{\pi}{3}\right)$, since the acquired speed of the particle and the tension of the string will both at first tend to diminish it.

Hence, since maxima and minima occur alternately, the reaction is a maximum at the lowest point, and a minimum when the string is inclined at an angle $\cos^{-1}\frac{5}{8}$.

2. Two particles, masses M and m ($M>m>M\cos\alpha$), are connected by a string passing over a smooth pulley; m hangs vertically, and M rests on a plane inclined at an angle α to the vertical. M starts without initial velocity from the point of the inclined plane vertically under the pulley. Show that M will oscillate through a distance

$$2m(M-m)h\cos\alpha/(m^2-M^2\cos^2\alpha),$$



where h is the height of the pulley above the initial position of M.

Let A be the initial position of M. M will evidently proceed down the plane. When it has proceeded a distance x, let m have been raised a vertical distance y. The tension of the string and the reaction of the plane do no work.

We may thus write the potential energy $-Mgx\cos\alpha+mgy$, if we choose the initial position as the standard position. Now, if at any point the potential energy has its initial value (zero), the kinetic energy will have its initial value, which is also zero. The condition for this is

$$Mgx\cos\alpha=mgy. \qquad \text{(1)}$$

Further, the length of the string between M and the pulley is $h+y$, and therefore

$$(h+y)^2=x^2+h^2+2hx\cos a,$$

or

$$y^2+2hy=x^2+2hx\cos a. \quad\text{.............................(2)}$$

From (1) and (2),

$$x^2\left(\frac{M^2\cos^2 a}{m^2}-1\right)=2hx\cos a\left(1-\frac{M}{m}\right),$$

whence $$x=0,$$

or $$x=\frac{2m(M-m)h\cos a}{m^2-M^2\cos^2 a}.$$

Denote this value of x by AB, in the figure.

For greater values of x than this latter the potential energy does not again vanish; it therefore keeps one sign, clearly the *positive*, since as x increases, x and y tend to equality, and $m > M\cos a$.

Hence the mass M cannot go beyond B, since the total energy is always zero, and the kinetic energy must be positive. It then returns up the plane, and the equation of energy shows that at A it is again instantaneously at rest, and the cycle of motions is repeated.

3. If a particle describes a closed path under conservative forces, prove that the curve of work cannot possibly be a closed oval curve, and that the algebraic sum of the areas included between it, the axis of x, and two terminal ordinates corresponding to any number of complete revolutions must be zero.

4. A heavy particle A is attached by a string of length a to a fixed point, and by a second string of length a to an equal particle B which slides on a smooth fixed vertical rod passing through the fixed point, and the system starts from rest with B below the fixed point, and the angle between the strings obtuse and equal to β. Prove that the speed of A in its lowest position will be

$$(6ag)^{\frac{1}{2}}\left(\cos\frac{\beta}{4}-\sin\frac{\beta}{4}\right).$$

5. To A and B, two fixed points on a vertical circle in the same horizontal line, the ends of an elastic string which passes through a bead on the circle are attached; the natural length of the string being AB and the bead being initially at A, find the pressure on the circle at any point.

6. Two equal particles A, B (of mass m) slide on a circular wire of radius a, which is fixed in a vertical plane, and are connected with a third particle C (of mass m') by two strings, each equal to the radius; the system starts from rest in the position in which the strings and the radii through A and B form a square, and C is vertically below the centre; find the speeds of the particles in

any subsequent position, and prove that when A and B meet, the speed of each is

$$\left\{(2-\sqrt{2})ag\frac{m+m'}{m}\right\}^{\frac{1}{2}}.$$

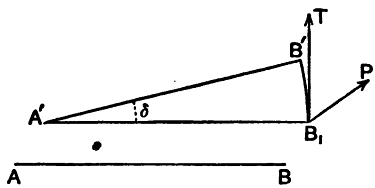
7. A small ring which can slide freely on a fixed vertical circle of radius a is connected with a particle of twice its mass by a string which passes over a pulley at one end of the horizontal diameter of the circle. If the ring be let fall from the other end of this diameter, show that its speed at the lowest point of the circle is

$$\sqrt{\{(5-2\sqrt{2})ag\}}.$$

118. *To find an expression for the* **work done by a couple** *of constant moment on a rigid body, rotated about an axis perpendicular to the plane of the couple.*

Two equal and parallel forces of opposite sense applied to different particles, but not in the direction of the line joining them, form a *couple.* If a is the length of the perpendicular distance between the lines of action of the forces, P the numerical value of either force, the product Pa is called the *moment* of the couple.

Let A, B represent two particles rigidly connected, to which the forces (P) of a couple are applied, and let A, B receive small displacements in the plane of the couple to A', B'. The forces



are to be regarded as constant during this displacement (§ 104), and we may therefore first move AB parallel to itself to the position $A'B_1$, and then rotate it round A' through a small angle whose circular measure is δ to the position $A'B'$. The work done in the former displacement is zero; that in the latter may (§ 108) be taken to be T (*arc* B_1B'), where T is the resolved part of P along the tangent at B_1 to this arc.

Now, $T(arc\, B_1B') = T\,.\,A'B_1\,.\,\delta = L\delta$, where L is the moment of the couple.

A similar expression holds for the work done in any number of small displacements. Hence if θ be the circular measure of

the angle between the initial and final positions of AB, the work done by the couple $= L\theta$.

The same expression evidently gives the work done by a couple applied to two particles A, B of a rigid body which is rotated about an axis perpendicular to the plane of a couple.

119. Equilibrium of a Particle under Conservative Forces. A particle placed without kinetic energy at a point where the potential energy (V) is a maximum or minimum will be in equilibrium, stable or unstable according to circumstances.

Definition of Stable Equilibrium. The equilibrium is said to be stable when the particle, if slightly displaced, never moves far from the position of equilibrium, and never acquires any great speed.

The typical cases which occur are as follows:

(1) If V is a minimum for all possible displacements, *i.e.* if the particle is not free to travel except to places of higher potential energy, the equilibrium is stable.

(2) If V is *constant* for all displacements, the particle will rest in all positions. The equilibrium is then called *neutral*.

(3) If V is a *maximum* for all possible displacements, the equilibrium is unstable.

(4) If V is a maximum for some displacements, and a minimum for others, the equilibrium is unstable for some displacements, stable for others.

In this case the equilibrium is *practically unstable*.

As an illustration of

(1) We may take a particle placed at the bottom of a smooth bowl and free to move along the surface.

(2) A particle placed at rest on a smooth horizontal table, and free to move along the surface.

(3) A particle placed at rest at the highest point of the outside surface of a smooth sphere and free to move along the surface.

(4) A particle placed at rest on a smooth saddle-backed surface.

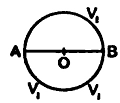
Here the equilibrium is stable for displacements along the length of the saddle, unstable for displacements across it.

The equilibrium is *practically* unstable.

A coin on edge is in neutral equilibrium for rolling motion, unstable for displacements perpendicular to its plane. The equilibrium is again practically unstable.

120. We will now prove these assertions for the important cases (1) and (3).

Let O be a point at which the potential energy has an absolute



maximum or minimum value V_0. Then the value of the potential energy in the former case decreases, and in the latter increases, along every path from O, at all events for a certain distance; we can therefore draw round O a small closed surface at every point of which the potential energy has a given value V_1, less than V_0 in the former, greater than V_0 in the latter case.

Draw any straight line through O, cutting this surface in A and B. By starting the particle from A with sufficient kinetic energy we can by means of smooth constraint (such as a smooth tube) compel it to travel along AB to B. The force on the particle as it passes O will clearly be the resolved part in the direction AB of the force which would act on a free particle placed at O.

But the kinetic energy of the particle will be the same at B as at A, since the potential energy is the same at these two points. Hence (§ 108) the *average force* from A to B is zero. Therefore, diminishing the surface indefinitely, the force at O resolved in the direction AB is zero; the same is true of the force in any other direction, and therefore O is a point of equilibrium.

Next place the particle at rest anywhere on the small closed surface. The energy of the particle is thus V_1. Then if $V_1 < V_0$, the particle can never move to the inside of the surface, but will move further away to places of lower potential energy, while if $V_1 > V_0$ it can never get outside the surface, and its kinetic energy can never exceed the small quantity $V_1 - V_0$.

Thus in the former case the equilibrium is unstable, in the latter stable.

Examples.

1. A bead is constrained to move along a smooth wire under the action of conservative forces. Prove the above proposition by considering the curve of work.

2. If the curve of work touches the axis of x at some point, but does not cut it there, what is the nature of the equilibrium of a

bead placed at the corresponding point of the wire? Illustrate from the case in which the only force acting is gravity.

3. Three particles of weights A, A, B, and three light strings of length $\frac{2\pi a}{3}$ connecting them, lie in a smooth circular tube of radius a; show that there are two possible positions of equilibrium—one stable, the other unstable; if the particles be placed in the unstable position and slightly displaced, find the greatest speed of their motion.

4. An elliptic wire is placed with its major axis $2a$ vertical and a bead on it is attached to the upper focus by an elastic string of natural length a; find the condition that the lowest position should be one of stable equilibrium.

121. Non-Conservative, or Dissipative, Forces. The characteristic of these forces as a class is that their magnitude and direction depend on the *velocities* of the particles to which they are applied. Such forces are the friction of solids sliding on solids, resistances due to viscosity of gases and liquids, to the induction of electric currents, and so forth.

The friction of a particle moving over a rough surface, for instance, has its *direction* determined by its velocity relative to the surface; the resistance of the air to a rifle bullet (regarded as a particle) is in direction opposite to the velocity and nearly proportional to a power of the speed between the second and third. Work done against such forces is not independent of the path, and the kinetic energy expended in work done against these forces disappears from the system as *mechanical energy*. The only forces of the non-conservative type that we shall have to discuss in this work are frictional forces, and thus, for our purposes, the equation of energy may be written

$$\text{kinetic energy} + \text{potential energy} + \text{work done against friction} = \text{constant}.$$

122. The general statement of the principle of the conservation of energy belongs to the domain of physics; the briefest outline must here suffice. We first generalise the definition of energy thus:

DEFINITION. *The energy of a system is its capacity for doing work.*

Now a system may be capable of doing mechanical work by virtue of the heat it contains, or by virtue of its gravitational, electrical, magnetic, chemical, or other condition; the system may therefore possess energy by virtue of any or all of these.

When work is done against resistance, mechanical energy disappears as such from the system; energy is also diminished when shocks and jars take place, as in machinery, and generally during any rapid alteration of stress; but heat is in every case evolved, and it has been shown by Joule and others that the heat evolved is a definite quantitative equivalent to the mechanical energy, kinetic or potential, which disappears; and further, that no energy is ever developed in (or lost from) a system without the disappearance (or appearance) of an equivalent amount, often different in form, elsewhere. Hence we may say:

The total energy of a material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms which energy is capable of assuming.

This is Clerk Maxwell's statement of the principle of the Conservation of Energy.

123. Forms of energy other than mechanical are attributed by physicists to the motion of the molecules of bodies, or to the motion of the ether. Heat, for instance, is supposed to be the kinetic energy of the motion of the molecules, each molecule in a solid body vibrating rapidly about a mean position. The attempt of physicists to apply the Third Law of Motion to molecules, atoms, etc., may or may not in the future be justified by the comparison of the results of theory and observation. Should this attempt be ultimately justified, we can understand how forces acting between all material parts of bodies may form a conservative system of the type discussed in § 115 (3). But when we consider the energy attributed to the motion of the ether, a further difficulty arises. The theorem of § 114 enables us to see how the mechanical, as distinguished from the physical, theory of the conservation of energy will apply to a system isolated in space, the only external forces acting on which are parallel forces applied to every particle of the system, the force applied to each particle being proportional to its mass. In fact, we have only to take the centre of mass of the system as kinetic origin. The inevitable loss of mechanical energy which, as experience tells us, such a system will undergo is supposed to be compensated by a transference of energy to the ether. Of the relation of the ether to the kinetic axes, however, nothing is as yet formulated, though we have evidence of the motion of the Earth relative to the ether in the phenomena of aberration. Now the measurement of the forms of energy attributed to the ether at present depends entirely on their equivalence to

mechanical energy estimated relative to the kinetic axes; it follows that the theory of the conservation of energy on this side cannot as yet be regarded as satisfactorily stated.

124. Power.

DEFINITION. *The power of an agent is the rate at which it can do work.*

Units of Power.

The British "absolute" unit is the power of an agent which does a foot-poundal in a second.

The C.G.S. unit is the power of an agent which does an erg in a second.

Both of these units are too small to be practically convenient.

Practical Units.

1. The *British practical unit* is the *horse-power*.

DEFINITION. The *horse-power* is the power of an agent which does 33,000 *foot-pounds* (foot-poundweights) of work in a minute, or 550 in a second.

2. The most important C.G.S. unit is the *watt*.

$$1 \text{ watt} = 10^7 \text{ ergs per second,}$$
$$1 \text{ watt} = \cdot 00134 \text{ horse-power,}$$
or
$$1 \text{ horse-power} = 746 \text{ watts.}$$

The horse-power of a steam engine is in practice found by means of the Indicator Diagram, which is a curve of the work done by the steam during one stroke of the cylinder, mechanically traced by an "indicator" temporarily attached to the engine. The indicated horse-power = equivalent of the area of the curve of work in foot-pounds × number of revolutions per minute ÷ 33000.

The "nominal horse-power" is only a number depending on certain measurements of the cylinders, etc.

Example. (1) *Find the horse-power of an engine which can drag a train on a level line with constant speed* v. *f.s., the resistance being equivalent to* R *pounds' weight.*

Since the train is running with constant speed, no work is expended in producing kinetic energy. The *only* work done is against the resistance. Now, in one second the train is pulled through v feet against the resistance.

$\therefore$ Rv foot-pounds of work are done in one second.

$\therefore$ the horse-power of the engine must be $\dfrac{Rv}{550}$.

Example. (2) *An engine exerts a constant force on a train of mass* m, *the resistance being equal to* R *poundals. Find the horse-power exerted by the engine at any instant of the motion.*

Let P be the force exerted by the engine in poundals, then if f be the constant acceleration of the train, we have, by the Second Law,
$$P-R=mf,$$
or
$$P=R+mf.$$

Now the work done per second is $P.v$ foot-poundals, which
$=(R+mf)v$ ... foot-poundals per second. ....(i.)

Hence the horse-power at which the engine is working when the speed of the train is $v=\dfrac{(R+mf)v}{550g}$.

Note. 1. An engine cannot exert *its full horse-power* on a train when *just starting*. For the horse-power put forth *increases with the speed*. Further, when the resistance and acceleration are constant, the power exerted is a maximum when the speed is a maximum.

2. The student acquainted with the elements of the Differential Calculus will see that the term Rv in formula (i.)

$$=\frac{d}{dt}(Rs)=\text{rate of work done against the resistance,}$$

while $\quad mvf=\dfrac{d}{dt}(\tfrac{1}{2}mv^2)=$ rate of increase of kinetic energy.

Example. (3) Assuming that work is worth 6d. per horse-power per hour, find the value of the energy stored up in the motion of a 5000 ton steamer moving at 20 miles an hour.

(4) What must be the resistance to a steamer of 1000 tons' displacement if the maximum speed attainable by means of an engine of 800 horse-power be 20 miles an hour?

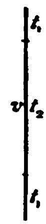
(5) An overshot waterwheel is turned by a waterfall 10 feet high and discharging 50 lbs. of water per second; it is found that, when the wheel is doing no useful work, 11 revolutions are made per second. Assuming that 10 per cent. of the energy of the falling water is used, that the only useless work done by the wheel is against the friction between the axle and its bearings, and that this friction is constant, find the useful work done per second when the wheel makes 5 revolutions per second.

Example. (6) *A cage of mass* m *is drawn up a mine by an engine, at first with uniform acceleration, then with uniform velocity* v, *and then for a distance equal to the first portion with uniform retardation; the whole time of ascent is* t, *and the greatest*

horse-power at which the engine is worked is H; *prove that the depth of the mine is*

$$\frac{500gHvt - mv^2(gt+v)}{550gH - mgv}.$$

Let f be the acceleration, t_1 the time taken to travel each of the end portions, t_2 the time taken to travel the middle portion.



Then $$t = 2t_1 + t_2.$$

The depth of the mine

$$= 2 \cdot \left(\frac{v+0}{2}\right) t_1 + vt_2 = v(t_1 + t_2)$$

$$= v(t_1 + t - 2t_1) = v(t - t_1)$$

$$= vt - \frac{v^2}{f}, \quad \text{.................................(i.)}$$

since $$v = ft_1.$$

The horse-power is greatest just before the velocity is uniform. Let T be the tension of the chain in *poundals* at this moment.

Then, by the Second Law,

$$T - mg = mf,$$

and the rate of work in absolute units $= Tv = m(g+f)v$.

$$\therefore \text{ the horse-power } H = \frac{m(g+f)v}{550g},$$

whence $$f = \frac{550gH}{mv} - g.$$

Substituting this value of f in (i.), we obtain the required result.

125. Horse-power transmitted by Belts and Shafts.

When a belt is turning a drum without slipping, it is easy to find an expression for the horse-power transmitted.

The receding part of the belt will be tighter than the approaching part. Let the tension of the former in *lbs.-weight* be T_1, and of the latter T_2. Also let v be the speed of the belt. Then in one second the receding part *does* $T_1 v$ foot-pounds of work; the approaching part has $T_2 v$ foot-pounds of work done on it.

$\therefore$ on the whole, $(T_1 - T_2)v$ foot-pounds of work are done per second by the belt, or the horse-power transmitted

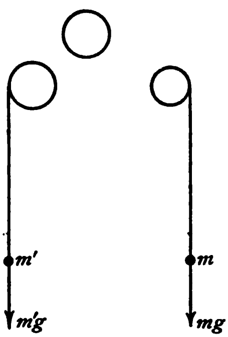
$$= \frac{(T_1 - T_2)v}{550}.$$

If a shaft is turning machinery, then if L be the moment of the couple (the units of force and length being 1 lb.-weight and a foot) turning the shaft, ω the angular velocity of the shaft in radians per second, $L\omega$ is the work done on the shaft per second in *foot-pounds*, and the horse-power transmitted is therefore $\frac{L\omega}{550}$.

A discussion of the practical conditions of the transmission of energy by a belt will be found in Garnett, *Elementary Dynamics*, §§ 137 β, 137 γ (ed. 1889), from which the preceding results are taken.

We add a few miscellaneous examples.

(1) *A system of pulleys without mass is such that the equilibrium "Power" and "Weight" are connected by the relation* W $=\mu$P. *If* W *and* P *are replaced by masses* m *and* m′ (m$>\mu$m′), *determine the motion of the system.*



When the equilibrium power and weight are attached, imagine P to move through any vertical distance "infinitely slowly" (see § 110 *supra*), so that the kinetic energy of the system is negligible. Then the algebraic sum of the quantities of work done by the external forces is zero.

$\therefore$ $P\times P$'s displacement
$= W\times W$'s displacement.

(The student will recognise here the statical principle of Virtual Work.)

Now this relation between the displacements is simply due to the geometrical connections of the system; hence the same relation will hold between the displacements when W and P are replaced by m and m'. Hence if the displacement of m is x and that of m' is y, we have

$$y=\mu x. \quad\text{.................................}(1)$$

If the system starts from rest, and if v, v' be the speeds of m, m' respectively after the displacements x, y, we have, equating the kinetic energy to the work done by the external forces,

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v'^2=mgx-m'gy.$$

And since $y=\mu x$ throughout the motion, it follows that $v'=\mu v$, whence

$$v^2=2\frac{m-\mu m'}{m+\mu^2m'}\cdot gx\ ;\quad v'^2=2\mu\,.\frac{(m-\mu m')}{m+\mu^2m'}\cdot gy,$$

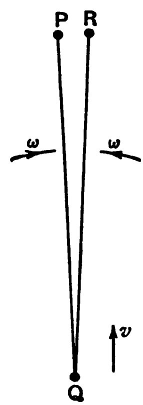
showing that the motion is uniformly accelerated, the acceleration of m being $\frac{m-\mu m'}{m+\mu^2 m'}\cdot g$, and that of m' being $\mu\cdot\frac{m-\mu m'}{m+\mu^2 m'}\cdot g$.

It should be noted that the method pursued in this problem does not necessitate the determination of the tensions.

The principles of the Conservation of Energy and of Momentum used simultaneously are often of great use in determining motions. In the next two examples the student will find illustrations of the use of the energy equation combined, in Example (2) with that of Linear, in Example (3) with that of Angular Momentum.

(2) *To each extremity and to the middle point of a string of length* 2a *is attached a particle of mass* m, *and the whole is laid in a straight line on a smooth horizontal table. Each of the end particles is then struck a blow* B *at right angles to the string. Show that at the instant when the two particles coincide the tension of the string is* $\frac{\mathrm{B}^2}{9\mathrm{am}}$.

Let v be the velocity of the middle particle Q just before the collision, ω the angular velocity of the two portions of the string at this instant.



Then the velocity of each of the end particles P, R relative to Q is at this instant numerically equal to $a\omega$ and at right angles to the common direction of the two portions of the string.

Since there are no external forces acting on the system, the kinetic energy remains unchanged and equal to its value just after the blows were struck.

Hence

$$\frac{1}{2}mv^2+2\cdot\frac{1}{2}m(v^2+a^2\omega^2)=2\cdot\frac{1}{2}\frac{B^2}{m}.$$

The linear momentum also remains unchanged.

Hence, taking its resolved part parallel to the direction of the blows, $3mv=2B$.

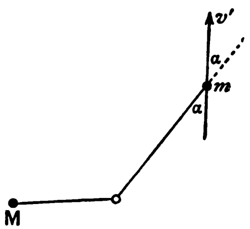
From these two equations $\omega=\frac{B}{am\sqrt{3}}$.

At the instant of meeting let f be the acceleration of Q, which is in the line of the strings in the sense QP or QR. The accelerations of both P and R relative to Q in the directions of the strings are each $a\omega^2$ in the opposite sense (see § 49). That of the centre of mass is zero.

Hence $$f+(f-a\omega^2)+(f-a\omega^2)=0,$$
or $$3f=2a\omega^2,$$
or $$f=\frac{2a\omega^2}{3}=\frac{2B^2}{9am^2}.$$
But if T be the tension, $2T=mf$.
$$\therefore \quad T=\frac{B^2}{9am}.$$

(3) *Two particles of masses* m *and* M *are connected by an inelastic string passing through a smooth fixed ring and are free to move on a smooth fixed horizontal plane through the ring. Initially the string is tight,* M *is at rest, and* m, *which is distant* l *from the ring, is projected with a velocity* v *in a direction perpendicular to the string between it and the ring. Prove that, when* m *is distant* r *from the ring, the string is passing through the ring with a speed*
$$\left\{\frac{mv^2}{m+M}\left(1-\frac{l^2}{r^2}\right)\right\}^{\frac{1}{2}}.$$

When m is distant r from the ring, let its velocity be v' inclined at an angle a to the string, and let that of M be V.



We thus have, since m is receding from the ring at the same rate as M is approaching it,
$$V=v'\cos a. \quad\text{.................(1)}$$

Since there is no external force acting on the system, the equation of energy is
$$\tfrac{1}{2}mv'^2+\tfrac{1}{2}MV^2=\tfrac{1}{2}mv^2. \quad\text{.........(2)}$$

The only force acting on m always passes through the point at which the ring is fixed; hence from the principle of the conservation of angular momentum (§ 94),
$$rv'\sin a=lv. \quad\text{...................................(3)}$$

From (1) and (3),
$$V^2+\frac{v^2l^2}{r^2}=v'^2,$$
and substituting for v' in (2), we have
$$V^2=\frac{mv^2}{M+m}\left(1-\frac{l^2}{r^2}\right).$$

(4) *A jet of water issues from a circular pipe of $\frac{1}{3}$ inch internal diameter at the rate of 4 gallons a minute; at what distance below the end of the pipe would the jet be capable of performing work with 1 horse-power if the whole of its energy could be used?*

A gallon of water weighs 10 lbs. and contains $277\frac{1}{4}$ cubic inches.

The section of the pipe $=\pi \cdot \frac{1}{6^2}=\frac{\pi}{36}$ square inches.

$\therefore$ speed of water on issuing

$$=\frac{4\times 277\frac{1}{4}}{\pi\times\frac{1}{36}}=\frac{1109\times 36\times 7}{22} \text{ inches per minute, taking } \pi=\tfrac{22}{7},$$

$$=\frac{1109\times 36\times 7}{22\times 60\times 12} \text{ feet a second}=\frac{1109\times 7}{440} \text{ feet a second.}$$

Now the mass of the water delivered per second $=\frac{40}{60}=\frac{2}{3}$ lb.

$\therefore$ kinetic energy supplied by this water per second

$$=\frac{1}{2}\cdot\frac{2}{3}\left(\frac{1109\times 7}{440}\right)^2.$$

Now let the depth at which the water can do 1 horse-power be h feet.

We have thus $\quad \frac{1}{2}\cdot\frac{2}{3}\left(\frac{1109\times 7}{440}\right)^2+\frac{2}{3}gh$

$=$energy of water per second in absolute units at depth h

$=550g$, since 1 horse-power $=550$ foot-pounds per second.

$$\therefore\ h=\frac{550\times 3}{2}-\frac{1}{2g}\left(\frac{1109\times 7}{440}\right)^2$$

$=(825-4{\cdot}86)$ ft. approximately.

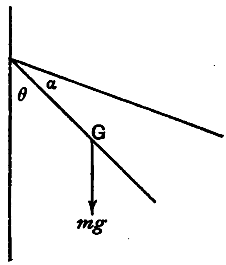
$=820{\cdot}14$ ft. approximately.

(5) *A rod is supported by a stiff joint at one end which will just hold it at an angle θ to the vertical. If the rod be lifted through a small angle (α) and then let go, prove that it will come to rest after moving through an angle $2\alpha-\frac{1}{3}\alpha^2\tan\theta$, approximately, supposing the friction couple at the joint to be constant.*

Let $2a$ be the length of the rod, m its mass, L the friction couple.

$$\therefore\ L=mga\sin\theta.$$

Now the rod when let go has no kinetic energy. Therefore when it is at rest again the work done by its weight must be equal to the work done against friction. Let ϕ be the angle the rod moves through before coming to rest.



The work done against friction is therefore $L\phi$ (§ 118).

Also the centre of mass of the rod moves through a distance

$$a[\cos\overline{\theta+a-\phi}-\cos\overline{\theta+a}].$$

$$\therefore\quad L\phi=mga(\cos\overline{\theta+a-\phi}-\cos\overline{\theta+a}),$$

or $$mga\sin\theta\,.\,\phi=mga(\cos\overline{\theta+a-\phi}-\cos\overline{\theta+a}),$$

or $$\phi\sin\theta=\cos(\theta+a-\phi)-\cos(\theta+a).\ \ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

$\therefore$ approximately

$$\phi\sin\theta=\cos(\theta+a)\left(1-\frac{\phi^2}{2}\right)+\sin(\theta+a)\left(\phi-\frac{\phi^3}{6}\right)-\cos(\theta+a).$$

$$\therefore\quad \sin\theta=-\frac{\phi}{2}\cos(\theta+a)+\sin(\theta+a)\left(1-\frac{\phi^2}{6}\right).$$

As a first approximation, neglect ϕ^2, ϕa and a^2.

$$\therefore\quad \sin\theta=-\frac{\phi}{2}\cos\theta+\sin\theta+\cos\theta\,.\,a.$$

$$\therefore\quad \phi=2a.$$

Now write $\phi=2a+x$ in equation (i.) and neglect squares of x.

$$\therefore\quad (2a+x)\sin\theta=\cos(\theta-a-x)-\cos(\theta+a)$$
$$=\cos(\theta-a)+x\sin(\theta-a)-\cos(\theta+a).$$

$$\therefore\quad x(\sin\theta-\sin\overline{\theta-a})=2\sin\theta\sin a-2a\sin\theta$$
$$=2\sin\theta\left(-\frac{a^3}{6}\right).$$

$$\therefore \quad x\cos\theta \,.\, a = 2\sin\theta\left(-\frac{a^3}{6}\right),$$

or

$$x = -\frac{a^2}{3}\tan\theta,$$

whence

$$\phi = 2a - \frac{a^2}{3}\tan\theta.$$

Examples on Chapter IV.

1. A particle of mass M hangs from a smooth vertical screw, and another particle of mass m is attached to the end of a horizontal arm rigidly connected with the screw; prove that, if the system is free to move, the acceleration of M is

$$\frac{p^2(M+m)g}{p^2(M+m)+4\pi^2a^2m},$$

where p is the distance between two consecutive threads, and a is the length of the arm, the inertia of the screw being negligible.

2. Two particles on a smooth horizontal table are attached by an elastic string of natural length a, and are initially at a distance a apart. One particle is projected at right angles to the string. Prove that, if the greatest length of the string during the subsequent motion be $2a$, the speed of projection is $\sqrt{\{8a\lambda/3m\}}$, where m is the harmonic mean between the masses of the particles, and λ is the modulus of elasticity of the string.

3. An inclined plane of angle α and mass M is free to move on a horizontal plane. Another plane of the same angle and of mass M' is laid on it so that its upper surface, on which is a particle of mass m, is horizontal. The surfaces are all smooth, and the motion takes place in a vertical plane. Show that the pressure of the particle m on the plane with which it is in contact is

$$MM'mg/\{MM'+(M+M')(m+M')\tan^2\alpha\}.$$

4. Two heavy particles of equal mass are connected by a weightless inelastic thin rod, and can move in a smooth circular tube fixed in a vertical plane, the tube having a narrow slit on its inner side so as to allow the rod to move freely in the plane of the circle without touching the tube. Prove that, if the rod initially occupies its upper horizontal position of equilibrium and is slightly displaced, then the resultant of the pressures on the tube at any subsequent time will be inclined to the rod at an angle

$$\tan^{-1}\{\cot\theta(3\cot^2\alpha+1)+2\cot^2\alpha\operatorname{cosec}\theta\},$$

where 2α is the angle subtended by the rod at the centre of the circle, and θ the inclination to the downward vertical of the radius vector from the centre of the circle to the middle point of the rod.

5. Two particles each of mass m are tied together by a light inextensible string of length $2a\alpha$, and are placed in a position of equilibrium in a vertical plane on the surface of a fixed smooth cylinder which has its axis horizontal and is of radius a. The particles are then slightly displaced. Prove that, when the lower of the particles in the subsequent motion leaves the surface, the pressure on the other is $2mg \sin \alpha \sin \theta$, where θ, the angular distance of the middle point of the string from the vertical, is determined by $2 \cos (\theta + \alpha) + \cos (\theta - \alpha) = 2 \cos \alpha$.

6. An empty box of mass M rests on a horizontal table, and from the centre of the lid swings a particle of mass m by a string of length l; the particle oscillates through a right angle on either side of the vertical. Prove that the box will not slide along the table if the coefficient of friction is greater than

$$\frac{3m}{2\{M(M+3m)\}^{\frac{1}{2}}},$$

but that if the table is smooth the box will oscillate through a range

$$\frac{2m}{M+m} \cdot l.$$

7. A railway truck is supported upon any even number of equal wheels whose weight may be neglected, ϵ is the angle of friction between each wheel and axle, and n is the ratio of the radii of each axle and wheel.

If the truck be started on the level with speed V, and if higher powers of n than the first be neglected, prove that if there be no sliding along the rails the distance it will go before it stops will be

$$\frac{V^2}{2gn \sin \epsilon}.$$

8. Two masses m_1, m_2 are connected by a weightless rod AB, and lie on a smooth horizontal table. The rod is struck at right angles to its length by a given blow I; find the velocities of the masses, and show that the kinetic energy is least if the point of application of the blow divides AB in the ratio $m_2 : m_1$.

9. Two particles whose masses are m_1 and m_2 are connected by a rigid rod whose mass may be neglected, and a third particle of mass M is tied by a light string to a point on the rod at distances a_1 and a_2 from m_1 and m_2 respectively, the whole resting on a smooth horizontal plane. M is projected along the plane and perpendicular to the rod with a velocity V. Show that just after the string becomes tight the kinetic energy of the particles is less than it was just before by the amount

$$\frac{MV^2}{2} \frac{m_1 m_2 (a_1 + a_2)^2}{M(m_1 a_1^2 + m_2 a_2^2) + m_1 m_2 (a_1 + a_2)^2}.$$

10. Two equally rough particles A and B of equal mass lie on a rough horizontal table. The particle A is attached by an inextensible string to a fixed point O in the table, and by another inextensible string to B. Initially OAB are in a straight line, A between O and B, and both strings tight. Then B is projected horizontally and perpendicularly to AB. Prove that, if A does not move before the string AB has turned through an angle

$$\cos^{-1}\left\{\frac{\sqrt{17}-1}{4}\right\},$$

it will not move at all.

11. A two-wheeled vehicle is being drawn along a level road with velocity v; the wheels (radius c) are connected by an axle (radius r) fixed to them, and the weight of the vehicle exclusive of the wheels and axle is W, and its centre of mass is vertically above the middle point of the axle. Show that, if the shafts are in a horizontal plane with the tops of the wheels, the horse is working at the rate

$$\frac{Wvr\sin\lambda}{\sqrt{(c^2-r^2\sin^2\lambda)}},$$

where λ is the angle of friction between the axle and its bearings.

12. Two weights W and W' balance on a system of pulleys. w is taken away from W' and added to W; find the acceleration of W, neglecting the mass of the system of pulleys.

13. A chain of length $2l$, in which the density increases uniformly from σ_1 at one end to σ_2 at the other, is hung over a small smooth pulley with equal lengths on each side, and set free. Show that the speed with which it leaves the pulley is

$$\left\{\frac{\sigma_1+5\sigma_2}{3(\sigma_1+\sigma_2)}gl\right\}^{\frac{1}{2}}.$$

14. Two rough-edged wheels of radii a, a' and of masses A, A' collected entirely on their rims, moving with angular velocities ω, ω', are placed with their edges in contact so that they are rotating round parallel fixed axes perpendicular to their planes. Show that the total work that can be lost in rubbing friction is

$$\frac{1}{2}\frac{AA'(a\omega\pm a'\omega')^2}{A+A'},$$

the ambiguity of sign depending upon the direction of ω with respect to that of ω'.

15. A smooth hemisphere of mass M whose plane surface slides on a horizontal table has a tube bored along its vertical radius in which slides a particle of mass Q, connected by a fine string with a particle of mass P which slides on the surface of the hemisphere.

Initially the system is at rest, with P in contact with the table and Q at the highest point of the tube. Show that, when P has described an arc $a\theta$ along the surface, the velocity of the hemisphere is

$$\sqrt{\frac{2ag(Q\theta - P\sin\theta)P^2\sin^2\theta}{(M+P+Q)[(M+P+Q)(P+Q)-(P^2\sin^2\theta)]}},$$

the length of string being $\frac{1}{4}$ of the circumference of the base of the hemisphere.

16. A bucket of mass M lbs. is raised from the bottom of a shaft of depth h feet by means of a light cord, which is wound on a wheel of mass m lbs. The wheel is driven by a constant force, which is applied tangentially at its rim for a certain time and then ceases. Prove that, if the bucket just comes to rest at the top of the shaft t seconds after the beginning of motion, the greatest rate of working in foot-poundals per second is

$$2ht\frac{M^2g^2}{Mgt^2 - 2h(m+M)}.$$

The mass of the wheel may be considered as condensed in its rim.

17. A smooth tube ABC of mass m is bent in one plane so as to be vertical at C and horizontal at B, which is vertically below A, the portion BC being a semicycloid of which C is the cusp, and the portion AB being of any shape without sharp angles. If the ends A and C slide on smooth horizontal wires in the plane ABC, AC preserving a constant direction, and a particle of mass $\dfrac{am}{h}$ starts from the upper end A when the tube is at rest and descends through the tube, show that while it moves through BC its speed along and relative to the tube is constant, h, a being the heights of A above B and B above C respectively.

18. A light rod with a mass P at one end and a mass Q at the other can move in a vertical plane, cutting at right angles the smooth planes AO, OB, which make respectively angles α, β with the horizon and support P and Q in such a manner that the rod does not pass through either plane. The rod is placed with P at O and PQ lying along OB; show that motion will take place if

$$Q\sin\beta\cos(\alpha+\beta)$$

is greater than $P\sin\alpha$, and find the speed with which Q will arrive at O.

19. $ABCD$ is a quadrilateral inscribed in a circle whose vertical diameter is AC. A heavy inelastic particle moves from rest at A along $ABCD$; show that in no case will its motion be continued

along DA; and if, after passing C, it first comes to rest indefinitely near D, prove that

$$\tan\phi = \frac{\sin 2\theta}{3 - \cos 2\theta},$$

where θ, ϕ are the angles subtended at A by BC and CD respectively.

20. Two particles M and m are connected by a string passing over a smooth pulley; the lesser mass m hangs vertically, and M is constrained to move in a smooth circular groove on a plane inclined at an angle α to the vertical, the highest point of the groove being vertically under the pulley. M starts without initial velocity from the highest point of the groove; show that, if it makes complete revolutions, the radius of the groove must be less than

$$m(M-m)h\cos\alpha/(m^2 - M^2\cos^2\alpha),$$

where h is the height of the pulley above the initial position of M.

21. A weightless rod has two particles of equal mass attached to its ends, and rests in a horizontal position on two perfectly rough pegs placed at its points of trisection. If the rod be set rocking, after how many impacts will the remaining energy be less than one-thousandth part of the energy originally imparted to it? Given

$$\log 2 = \cdot 3010300.$$

22. Two small rings of equal masses attached together by a straight string slide on a smooth vertical circular wire; show that, however they be projected, the string will not remain tight unless its middle point lies in the upper half of the circle.

23. A smooth cylinder is held with its axis horizontal. Two masses M, m connected by an inextensible string are hung over the cylinder in a plane perpendicular to the axis, m touching the cylinder in a level with its axis, and M hanging freely. If M descends under the action of gravity, prove that m will leave the cylinder before it reaches the highest point if $\dfrac{M}{m} > 1\cdot 401$.

24. A string has attached to its extremities masses each equal to M, and is then passed over two smooth pulleys whose centres are in a horizontal line, and at a distance apart $=2a$. A mass $2M$ is attached midway between the pulleys and is then let go; show that the subsequent speed of the mass $2M$

$$= 2\sqrt{\frac{ga\left(1 - \tan\frac{\phi}{4}\right)}{3 + \cos\phi}},$$

where ϕ is the angle between the two parts of the string.

25. A wedge of mass M can slide on a smooth horizontal plane. The wedge has a smooth face inclined at an angle α to the horizon. Initially the wedge is at rest; and a particle of mass m is projected directly up the inclined face. Prove that, if the particle rises to a height h above its point of projection, its speed of projection is

$$\left\{2gh\frac{M+m}{M+m\sin^2\alpha}\right\}^{\frac{1}{2}}.$$

26. Two masses m_1, m_2 at A and B are connected by a rigid light rod. The rod is struck by a given impulse P at right angles to AB at the centre of gravity of m_1, m_2. Show that the kinetic energy generated is

$$\frac{1}{2}\frac{P^2}{(m_1+m_2)}.$$

27. A mass m is attached to another mass m' by an inextensible string which is straight, and a blow P making an angle α with the line of the string is applied to m; prove that the work done is

$$\frac{1}{2}\cdot\frac{P^2\cos^2\alpha}{m+m'}+\frac{1}{2}\cdot\frac{P^2\sin^2\alpha}{m},$$

and that the impulsive tension of the string is

$$\frac{m'}{m+m'}\cdot P\cos\alpha.$$

28. A shot of mass m lbs. just penetrates through a plate of M lbs. and thickness t, which is free to mòve; show that it would penetrate a thickness $(1+m/M)t$ of the same material held fixed if projected with the same velocity.

29. A, B are two smooth holes in a smooth horizontal table, their distance apart being $2a$. A particle of mass M rests on the table midway between A and B, and a particle of mass m hangs beneath the table, suspended from M by two equal weightless and inextensible strings passing through the two holes. The length of each string is $a(1+\sec\alpha)$. A blow J is applied to M in a direction perpendicular to AB. Show that if $J^2 > 2Mmag\tan\alpha$, M will oscillate to and fro through a distance $2a\tan\alpha$, but that if J^2 is less than this quantity and $=2Mmag(\tan\alpha-\tan\beta)$, the distance through which M oscillates will be

$$2a\,.\sqrt{(\sec\alpha-\sec\beta)(\sec\alpha-\sec\beta+2)}.$$

30. A particle slides down the outside of a smooth parabola with axis horizontal, starting from rest at a point whose ordinate is k; prove that the ordinate of the point where it leaves the curve is the positive root of the equation

$$y^3+12a^2y-8a^2k=0,$$

where $4a$ is the latus rectum.

31. A string of length l is attached to a particle of mass m, which slides on a rough horizontal plane, the other end of the string being attached to a fixed point in the plane. Initially the string is just tight, and the particle is projected with velocity u at right angles to the string. Find the whole space described by the particle before coming to rest, and prove that the tension of the string when the particle has described a space s is equal to

$$\frac{m}{l}(u^2 - 2\mu g s),$$

where μ is the coefficient of friction.

32. Apply the methods of energy and conservation of angular momentum to the following problem. A particle which can move on a smooth horizontal table is attached to a fixed point of the table by an elastic unstretched string of length a, which is such that the weight of the particle would stretch it to twice its natural length. Prove that, if the particle be projected with velocity u perpendicular to the string, its speed v when next moving at right angles to the string will be given by

$$v^3 + v^2 u + vga - uga = 0.$$

33. A hoop of radius r, without mass, has attached to it at the ends of a diameter two particles of mass m. The hoop starts to roll from rest down an inclined plane sufficiently rough to prevent slipping. Find the speed of its centre after it has descended a length l of the plane.

34. Two wheels of radii R_1 and R_2 respectively can slip on their axles, which are rough. The wheels are connected by a perfectly rough band passing round both of them, their axles being parallel. If the two axles rotate with uniform angular velocities ω_1 and ω_2, and the wheels slip slightly on their axles, but so that the band travels with uniform speed, show that the horse-power with which the second axle is working will be to the horse-power exerted on the first axle in the ratio $R_2\omega_2 : R_1\omega_1$.

35. A train running at a full speed of 30 miles per hour slips two carriages, each of mass 10 tons, and the full speed rises to $33\frac{1}{3}$ miles per hour; find the horse-power of the engine and the mass of the train, the resistance due to friction, etc., being 12·5 lbs. per ton.

36. One engine A starting from rest generates in two minutes in a train a velocity of 45 m.h. while it passes over a distance of 1 mile on the level. Another engine B of equal weight can pull the same train up an incline of $\sin^{-1}\frac{1}{80}$ at a "full speed" of 20 m.h. Assuming that the resistance due to friction, etc., is constant and equal to the weight of 12 lbs. per ton, prove that the time-average of the

horse-power at which A works for the two minutes is 1·53 ... times the horse-power of B.

37. A Nasmyth hammer is worked by steam pressure on a piston whose area is four square feet, and whose mass together with that of the hammer is ten tons. The excess of pressure on the upper side of the piston is sixty pounds per square inch, and the hammer after descending through two feet strikes a mass of iron, and is brought to rest after compressing it through half an inch. If the resistance of the iron to compression be uniform, find its amount, and the work done by the steam.

38. An engine works at a constant rate, and changes the speed of a train from u to v in T seconds. Neglecting the resistance due to friction, find the speed after t seconds, the acceleration at that time, and (when $u=0$) the distance covered in t seconds.

39. A train whose mass is 150 tons has an engine of 230 horse-power. Find the greatest uniform speed that can be maintained while ascending an incline of 1 in 80, the resistance being equal to the weight of 1 ton.

40. If the level of the water in docks covering 100 acres be altered 20 ft. by the tide twice every 25 hours; find the horse-power of an engine which could be driven by the energy, if it could all be made use of. (Weight of 1 cubic foot of water $=1000$ ozs.)

41. A fire engine pumps water from a well, and discharges it with a velocity of 50 f.s. through the orifice of a pipe whose section is 1 sq. in., and which is at a height of 25 feet above the surface of the water in the well. Find at what horse-power the engine is working. (Weight of 1 cubic foot of water $=1000$ ozs.)

42. The horse-power of three engines forming a train are H_1, H_2, H_3 in order from the front of the train, and $H_1 > H_2 > H_3$. Given the masses of the engines and the retarding force due to friction, etc., for each lb.; find the tensions of the couplings when the train is at full speed.

43. A train of mass m runs from rest at one station to stop at the next at a distance l. The full speed is V and the average speed is v. The resistance of the rails when the brake is not applied is uV/lg of the weight of the train, and when the brake is applied it is $u'V/lg$ of the weight of the train. The pull of the engine has one constant value when the train is starting, and another when it runs at full speed. Prove that the average rate at which the engine works in starting the train is

$$\frac{1}{2}m\frac{V^2}{l}\left[u+\frac{1}{\frac{2}{v}-\frac{2}{V}-\frac{1}{u'}}\right].$$

44. An engine of mass M tons, when working at horse-power H, draws n carriages, each of mass M' tons, at the uniform rate of v miles an hour. Supposing the resistance on the engine and on each carriage to be proportional to the weight, prove that the tension of the coupling between the engine and the nearest carriage is equal to the weight of

$$\frac{75}{448} \cdot \frac{HnM'}{(M+nM')v} \cdot \text{tons.}$$

45. Prove that the extra work required to take a train from one station to stop at the next at a distance l ft. in time t seconds is

$$\frac{l}{\frac{1}{8}gt^2} \cdot \frac{1}{k} \Big/ \left\{ \left(\frac{1}{m}+\frac{1}{n}\right)\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{k}\right) \right\}$$

times the work required to run through with uniform speed without stopping, where the incline of the road is 1 in m and the resistance of the road and the brake power per unit mass are equal to the component of gravity down uniform inclines of 1 in n and 1 in k respectively.

46. Power is transmitted from one shaft to another by means of a single open belt, 6 in. wide and $\frac{1}{4}$ in. thick, running at a speed of 60 ft. per second. If the tension in the loose side of the belt is one-half that in the tight side, and a maximum stress of 300 lbs. per square inch be allowed, find, neglecting the weight of the belt, what H.P. may be transmitted.

CHAPTER V.

UNITS AND DIMENSIONS.

126. WE have already explained that to measure any quantity we require (1) a unit, (2) a number expressing the ratio of the quantity to the unit.

We have found it convenient to distinguish the units of length, time, and mass as *fundamental* units. We have derived the unit of each complex quantity from these in as simple a manner as possible; the system of units so formed is called the absolute system, a name taken from a paper by Gauss in which the unit of force, as derived in § 63, is defined.* The simplicity of this system greatly assists the memory.

We have throughout this work confined ourselves to two particular systems of fundamental units, viz., the British and the C.G.S. units. But the use of these is not compulsory nor even always convenient.

One object of this chapter is to show how to change rapidly from any one system to another. We shall find it convenient to call one arbitrarily chosen system of units the *Standard System*; a system into which the change is supposed to be made will be called the *New System*. In any particular case any convenient system may be regarded as the standard.

127. Dimensions. Let the unit of

mass in the new system be M standard units,
length " " L "
time " " T "

So that M, L, T are pure numbers, viz., the ratios of the respective pairs of units in the new and standard system.

Then any derived unit of the new system

$= M^x L^y T^z$ corresponding units of the standard system.

* *Intensitas Vis Magneticae Terrestris ad Mensuram Absolutam Revocata.* Werke, Vol. V., p. 85.

For example, suppose that the standard system is the foot-pound-second system, and that the fundamental units of the new system are M lbs., L ft., and T seconds: then the new unit of velocity is a velocity of L feet per T seconds, that is $\frac{L}{T}$ feet per second; writing $\frac{L}{T}$ as LT^{-1}, we see that in this case $x=0$, $y=1$, $z=-1$. Similarly the new unit of acceleration is an acceleration which gives an increase of $\frac{L}{T}$ units of velocity in T seconds; its magnitude therefore is $\left(\frac{L}{T} \div T\right)$ f.s.s., or LT^{-2} f.s.s.; in this case $x=0, y=1, z=-2$. The new unit of force is MLT^{-2} poundals, and in this case $x=1$, $y=1$, $z=-2$.

M, L and T are numbers which will depend on the particular systems employed, but the values of x, y, z are the same for the same unit whatever systems are employed. The numbers x, y, z are called the *dimensions* of that unit, or the dimensions of the quantity which is measured in terms of the unit. The latter nomenclature conduces to brevity of statement.

The symbol $M^xL^yT^z$ may be called the **dimension-symbol** of the unit.

Thus the dimension-symbol is the measure of the new unit in terms of the corresponding unit of the standard system.

A quantity which has the dimension-symbol $M^xL^yT^z$ is said to be of x dimensions in mass, y in length, z in time.

For instance, the dimension-symbol of a velocity is $M^0L^1T^{-1}$, or LT^{-1}; or the dimensions of a velocity are 0 in mass, 1 in length, -1 in time. That of an acceleration is LT^{-2}; or the dimensions of an acceleration are 0 in mass, 1 in length, -2 in time, and so on. The dimension-symbol being merely an algebraic product, these dimension-symbols may, if we choose, be written $\frac{L}{T}$, $\frac{L}{T^2}$.

128. When a quantity A is so defined that its measure is the product of the measures of several other quantities B, C, D, ... the dimension-symbol of the quantity A is the product of the dimension-symbols of the quantities B, C, D.

For instance, the dimension-symbol of a mass is M, that of a velocity is LT^{-1}, that of a momentum (measure mv) is MLT^{-1}.

In a *physical equation* the complete terms represent numbers all of which refer to one and the same unit; consequently the dimensions of every term must be the same.

For instance, in the energy equation of a body projected vertically upwards (§ 116),

$$\tfrac{1}{2}mv^2 + mgh = \tfrac{1}{2}mu^2, \text{ ....(i.)}$$

the dimension-symbols of the terms are, in order,

$$M \cdot \overline{\frac{L}{T}}\Big|^2, \quad M \cdot \frac{L}{T^2} \cdot L, \quad M \cdot \overline{\frac{L}{T}}\Big|^2,$$

all of which are the same.

This property affords a valuable method for verifying physical equations; any want of homogeneity must be the result of error.

For instance, the quantities having the same meaning as in (i.), $mvh + m^2v^2 = 2mg$ would not be a possible equation.

In estimating the dimensions of a quantity it should be noted that the measure of a mass may be any homogeneous function of degree unity in numbers representing masses, and of degree zero in numbers representing lengths or times, a similar remark being true of the measures of a length and time.

Thus $\dfrac{mm'}{m+m'}$ or $\dfrac{am+bm'}{a+b}$, m, m' being the measures of masses and a, b of lengths, may be measures of a mass, and the quantities they then represent have consequently the dimension-symbol ML^0T^0.

129. For convenience we append here the dimension-symbols of the commoner derived quantities; the student will have no difficulty in verifying them.

Velocity, $\dfrac{L}{T}$. Acceleration, $\dfrac{L}{T^2}$.

Circular measure of angle, L^0. Angular velocity, T^{-1}.

Momentum or impulse, $\dfrac{ML}{T}$. Angular momentum, $\dfrac{ML^2}{T}$.

Mass-acceleration or force, $\dfrac{ML}{T^2}$.

Work or energy, $\dfrac{ML^2}{T^2}$. Power, $\dfrac{ML^2}{T^3}$.

Density (mass per volume), $\dfrac{M}{L^3}$. (See § 137.)

Examples. If m denote the measure of a mass, l of a length, t of a time, v of a velocity, f of an acceleration, P of a force, which of the following equations are possible ones?—

(i.) $mft - mv = Pt.$

(ii.) $\frac{1}{2}mv^2 + mfs = Pv.$

(iii.) $mvf + \frac{Pl}{t} = \frac{mv^2}{t}.$

(iv.) $(m+m')vt = ml + l\sqrt{m^2 - m(m+m')\frac{a}{b}}$, a and b denoting lengths.

130. Let q, q' be the measures of a given quantity in two different systems of units, and let the unit of the quantity contain in the first system U, and in the second system U' units of the standard system. Then the measure of the quantity in the standard system is either qU or $q'U'$; consequently

$$qU = q'U',$$

or *the measure of a quantity varies inversely as the measure of its unit expressed in the standard system.*

131. The equation just obtained is of great importance in the transformation of quantities from one system to another.

Let x, y, z be the dimensions of the quantity whose measures in the two systems are respectively q and q'.

Also let $\left.\begin{matrix} M, L, T \\ M', L', T' \end{matrix}\right\}$ be the respective measures of the fundamental units of the two systems in *any convenient system* which we select as standard, then since

$$U = M^x L^y T^z, \quad U' = M'^x L'^y T'^z,$$

we have $$q \,.\, M^x L^y T^z = q' \,.\, M'^x L'^y T'^z,$$

and q is at once determined when q' is known.

We may of course choose one of the two systems themselves as the standard system, which amounts to putting $M' = 1$, $L' = 1$, $T' = 1$ in the above equation.

Example 1. *Given that* 1 *gram* = ·0022 *lbs. and* 1 *centimetre* = ·4 *inches approximately, express the foot-pound in ergs.*

The equation for transforming measures of work is

$$q \cdot \frac{ML^2}{T^2} = q' \cdot \frac{M'L'^2}{T'^2}.$$

Taking the foot-pound-second system as standard, we may write $q' = g$, $M' = L' = T' = 1$, $M = \cdot 0022$, $L = \frac{\cdot 4}{12} = \frac{1}{30}$, $T = 1$; and

q, the number of ergs in a foot-pound, is given by

$$q\,.\,{\cdot}0022\,.\,\overline{\frac{1}{30}}\Big|^2=g,$$

or $$q=\frac{32\times 30\times 30}{{\cdot}0022}=13\times 10^6 \text{ ergs approximately.}$$

Example 2. *If the unit of acceleration be that due to gravity, the unit of mass* 1 *pound, and the unit of power* 1 *horse-power, find what number will express a velocity of* 1 *foot per second.*

If M, L, T be the new units of mass, length, and time, we have from the above data, taking the foot-pound-second system as standard,

$$\frac{L}{T^2}=g,\quad M=1,$$

$\frac{ML^2}{T^3}$ = measure of 1 horse-power in absolute units in the British system $=550g$, whence $\frac{L}{T}=550$, or the new unit of velocity $=550$ ft. per second.

Taking x for the measure of the velocity in the new system, we have

$$x\times 550=1\times 1,$$

or $$x=\tfrac{1}{550}.$$

132. By a consideration of dimensions very many formulae may be to some extent predicted. Thus, supposing that we have observed that the vibrations of a simple pendulum (which consists of a particle tied by a light string to a fixed point) are independent of the *amplitude*, as is the case for small vibrations, we can determine the manner in which the time of vibration depends on the various quantities involved.

For let m be the mass of the particle,
l be the length of the string,
g be the value of the acceleration due to gravity.

Then the time of vibration can depend on these alone, and we may assume t the time to consist of a series of terms of which $Am^xl^yg^z$ is the type, A being a *number*.

Now the *dimension-symbol* of this quantity is

$$M^xL^y\,.\,\overline{\frac{L}{T^2}}\Big|^z=M^xL^{y+z}T^{-2z},$$

and this is to have the same dimensions as a *time*;

$$\therefore\;\; x=0,\quad y+z=0,\quad -2z=1.$$

To these there is but one solution, and therefore only *one term* in the expression for the time;

$$\therefore\ y=\tfrac{1}{2},\quad z=-\tfrac{1}{2},$$

and the time $\propto\sqrt{\dfrac{l}{g}}.$

As a fact the time of a complete vibration

$$=2\pi\sqrt{\frac{l}{g}}.$$

As another example we may take the following:

The frequency (n) *of vibration of a violin string of given length depends on* c *the cross-section of the string*, m *its mass per unit length, and* t *its tension. Prove that* $n^2\propto\dfrac{t}{mc}$.

n, the frequency, is defined as the number of vibrations taking place in a second, and has therefore the dimension-symbol $\dfrac{1}{T}$.

Further, we may write $n=\Sigma Am^x t^y c^z$, where A is a number.

$Am^x t^y c^z$ has for dimension-symbol,

$$\left|\frac{M}{L}\right|^x\cdot\left|\frac{ML}{T^2}\right|^y\cdot L^{2z}.$$

Whence $\quad x+y=0,\ -x+y+2z=0,\ -2y=-1,$

or $\quad y=\tfrac{1}{2},\ x=-\tfrac{1}{2},\ z=-\tfrac{1}{2}.$

The solution is again unique, and therefore

$$n\propto\left\{\frac{t}{mc}\right\}^{\frac{1}{2}},$$

or

$$n^2\propto\frac{t}{mc}.$$

133. Universal Gravitation. Newton's Law of Universal Gravitation asserts that when two particles of masses m and m' respectively are at a distance r apart, m produces in m' an acceleration numerically equal to $C\cdot\dfrac{m}{r^2}$, while m' produces in m an acceleration numerically equal to $C\cdot\dfrac{m'}{r^2}$, these accelerations being in the line joining the particles, and the sense of each acceleration being towards the particle which produces it. C is

a constant called the constant of gravitation, whose value depends on the particular units employed.

The magnitude of the reaction between the particles is thus

$$C \cdot \frac{mm'}{r^2}.$$

It is to be understood that these accelerations are estimated relative to the kinetic axes, the kinetic origin being the centre of mass of the solar system (to which for many purposes the centre of the Sun is a sufficiently close approximation), and the directions of the axes being determined by reference to the fixed stars. The law also is considered to hold approximately for two particles near the Earth's surface if the motion of the particles is referred to Galileo's axes; thus applied, it forms the basis of the Cavendish Experiment, for the details of which see Maxwell, *Matter and Motion*, Articles CXLII.-CXLIV.

It can be shown that a homogeneous sphere accelerates external masses in the same way as a particle, whose mass is equal to that of the sphere, would do if placed at the centre of the sphere. This we shall assume.

134. To determine the Value of the Constant of Gravitation in the C.G.S. System. The acceleration produced by the mass of the Earth in a particle near its surface is 981 cm.s.s. Regarding the Earth as a homogeneous sphere and neglecting the effect of its rotation, this acceleration is in the direction of, and relative to, the Earth's centre; relative to the common centre of mass of the Earth and the particle it will be $981 \frac{E}{E+m}$ (§ 85), where E is the mass of the Earth, and m that of the particle, in grams. Now, $E = 6{\cdot}14 \times 10^{27}$ grams, a very large quantity compared with the mass of the particle; hence the acceleration of the particle may be taken as 981 cm.s.s. *relative to the common centre of mass.*

Since all the external forces on the particles of which the Earth is composed, including particles free to move near its surface, are sensibly parallel and proportional to the mass of each particle, we may, by § 88, take the common centre of mass as kinetic origin.

The acceleration of the particle is therefore by the law numerically equal to $C \cdot \frac{E}{R^2}$, where R is the Earth's radius in centimetres.

Hence $$C \cdot \frac{E}{R^2} = 981.$$

But $\qquad R = 6{\cdot}37 \times 10^8$ approximately;

$$\therefore \quad C = 981 \times \frac{(6{\cdot}37 \times 10^8)^2}{6{\cdot}14 \times 10^{27}}$$
$$= 6{\cdot}48 \times 10^{-8}.$$

135. We may also so determine the unit of mass that the constant C may be unity; the unit of mass so determined is called the **astronomical unit of mass**; if $C = 1$, the acceleration produced by a particle of mass m on any particle at distance r is numerically equal to $\frac{m}{r^2}$; putting $m = 1$, $r = 1$, this acceleration is unity; hence the following definition:

The astronomical unit of mass is the mass of a particle which produces unit acceleration in any other particle placed at unit distance from it.

The force between two particles of masses m and m' placed at a distance r apart is now $\frac{mm'}{r^2}$; hence two particles of unit mass placed at unit distance apart attract each other with unit force.

To determine the astronomical unit of mass, and the corresponding unit of force, when the units of length and time are one centimetre and one second.

Let the new unit of mass contain x grams. If the mass of the Earth be E grams, its mass in terms of the new unit is $\frac{E}{x}$.

Hence the acceleration produced by it at distance R equal to the Earth's radius $= C \cdot \dfrac{\frac{E}{x}}{R^2} = \dfrac{E}{xR^2}$, since by hypothesis $C = 1$.

Therefore, as in § 134, $\dfrac{E}{xR^2} = 981$,

or $$x = \frac{E}{981 R^2} = 1{\cdot}543 \times 10^7 \text{ grams approximately.}$$

The unit force thus produces an acceleration unity in a mass of $1{\cdot}543 \times 10^7$ grams. Its value is therefore $1{\cdot}543 \times 10^7$ dynes.

136. Dimensions of Mass when the Astronomical System of Units is employed. Since the symbol $\frac{m}{r^2}$ represents an acceleration, it is of dimensions $\frac{L}{T^2}$; it follows that m must be

of dimensions $\frac{L^3}{T^2}$. This means that if given units of length and time and the corresponding derived astronomical unit of mass be chosen as the standard system, then the astronomical unit of mass in a new system, in which the units of length and time are respectively L and T standard units, will be $\frac{L^3}{T^2}$ standard units.

This, of course, is merely the result of our definition of mass-ratio as an inverse ratio of accelerations, combined with the Newtonian law of inverse squares. To make this clear, we repeat the above argument in a slightly different form. If m, m' be the measures of two masses, f, f' the *numerical* values of their mutually induced accelerations, then $mf=m'f'$, or $m=m'.\frac{f'}{f}$.

Now let m' be the unit mass of the standard system. Consequently, employing this system, $m'=1$, $f=\frac{1}{r^2}$; so that the measure of any mass m is $f'r^2$, a product whose dimension-symbol is $\frac{L}{T^2}.L^2$, or $\frac{L^3}{T^2}$.

137. We have not hitherto had occasion for the use of the conception of **density**.

DEFINITION. *The density of a homogeneous substance is its mass per unit volume.*

When a substance is not homogeneous, its density at any point is the limit to which the ratio $\frac{m}{V}$ tends, where m is the mass of a portion of the substance of volume V enclosing the point, when V is diminished indefinitely.

The unit of density in the C.G.S. system is that of a substance, one cubic centimetre of which has the mass of one gram. In other words, it is very nearly the maximum density of distilled water, which occurs when the water has a temperature of 4° C. (See Definition of the Gram, § 60.)

There is no convenient homogeneous substance whose density is that of 1 lb. to the cubic foot. Hence it is usual in the British system to *compare* the density of a substance with that of distilled water at its maximum density. The ratio is then often called the density of the substance (water unity); the correct name for this ratio is the *Specific Gravity* of the substance.

The dimension-symbol of density is $\frac{M}{L^3}$.

Specific gravity, however, being a *ratio* of two densities, is of *zero* dimensions in mass, length, and time.

The dimension-symbol of density when the astronomical unit of mass is employed is therefore $\frac{L^3}{T^2} \div L^3$, or $\frac{1}{T^2}$, and is independent of the unit of length.

The curious result just arrived at is susceptible of an easy physical interpretation.

Conceive a spherical planet of radius r, density d, and imagine a satellite to revolve in a circular orbit close to its surface in time t. Let the mass of the satellite be so small that the centre of the planet may be taken as kinetic origin.

The mass of the planet is $\frac{4}{3}\pi r^3 d$.

Hence, by the law of gravitation, if we employ astronomical units, the acceleration of the satellite to the planet's centre is

$$\frac{\frac{4}{3}\pi r^3 d}{r^2} = \frac{4}{3}\pi r d.$$

But this acceleration is also equal to $\left(\frac{2\pi}{t}\right)^2 . r$, since $\frac{2\pi}{t}$ is the satellite's angular velocity.

Hence $\frac{4}{3}\pi r d = \frac{4\pi^2}{t^2} . r$, or $d = \frac{3\pi}{t^2}$.

The dimensions of d are therefore merely -2 in time.

Examples on Chapter V.

1. If the unit of work be the work done in lifting 1 cwt. through 3 yards, and the unit of momentum that of a mass of 1 lb. which has fallen vertically through 4 feet from rest under the action of gravity, and the unit of acceleration three times that produced by gravity in a falling body, find the units of space, time, and mass.

2. If the unit of work be a foot-pound, the unit of momentum that of a body whose mass is an ounce after it has fallen through g feet, and the unit of power 33,000 foot-pounds of work per minute, find the units of mass, length, and time.

3. Compare the unit of energy when the fundamental units are a feet, b pounds, and c seconds, with its value referred to 1 foot, 1 pound, and 1 second.

Find the kinetic energy in both systems of the Earth, assuming it to be a sphere of 4000 miles' radius of mean density $5\frac{1}{2}$ (water unity), and the distance of the Sun to be constant and equal to 93,000,000 miles; neglecting the rotation of the Earth on its axis, and assuming the weight of a cubic foot of water to be 1000 ounces.

4. A bicycle, whose mass together with that of its rider is p pounds, is being ridden m miles an hour, and the resistance is equal to the weight of r pounds: if the power of the rider be denoted by

x, his momentum by y, and the resistance by z, find the units of mass, length, and time in terms of the pound, foot, and second.

5. If the unit acceleration be that of a body falling freely, the unit of velocity the velocity acquired by the body in a seconds from rest, and the unit of momentum that of 1 lb. after falling for b seconds, determine the units of length, time, and mass.

6. A light cord is attached to a point in the circumference of a fly-wheel, whose mass is uniformly distributed round its circumference, and is coiled round the wheel. A bucket, whose mass is one quarter of that of the wheel, is attached to the end of the cord, and just rests at the bottom of a shaft. If unit impulse applied to the rim of the wheel will compel it to commence rotating at a certain rate per second, and if unit rate of doing work is such as to maintain this rate, show that ten times the number of feet in the unit of length is equal to the square of eight times the number of seconds in the unit of time.

7. If the momentum of a train weighing 50 tons and moving with a velocity of 15 m.h. be denoted by 1 and its kinetic energy by 5, and if the force necessary to give it an acceleration (if unresisted) of 5 f.s.s. be denoted by 100, show that the measure of the acceleration due to gravity is 6400.

8. Prove from considerations of "dimensions" that the time a particle will take to fall from rest at a great distance D to the Earth's surface will vary as $D^{\frac{3}{2}}$.

9. Assuming the satellites of Mars to describe circular orbits round the centre of the planet, find the time of revolution of the second satellite, it being known that the time of revolution of the first is 7 h. 40 m., and that their distances from the centre of the planet are in the ratio of 2 : 5 approximately.

10. If the value of the "constant of gravitation" is $6{\cdot}6576 \times 10^{-8}$, find the attraction in dynes between two spheres of water, each one metre in diameter and just touching.

11. Show that if a particle be revolving with uniform angular velocity in a circle of radius 1 cm. under the attraction of a small fixed sphere of mass 10 grams concentric with the circle, the period of revolution is roughly $2\frac{1}{6}$ hours.

[Mass of Earth $= 6{\cdot}14 \times 10^{27}$ grams.
Radius of Earth $= 6{\cdot}37 \times 10^{8}$ cm.
Weight of 1 gram $= 981$ dynes.]

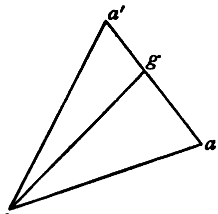
12. Prove that the dimensions of force and power, when the astronomical unit of mass is employed, are respectively those of the fourth and fifth powers of a velocity.

Illustrate the dimensions of force by considering the case of two equal gravitating masses revolving in circular orbits about their common centre of mass.

CHAPTER VI.

SPECIAL PROBLEMS. IMPACT.

138. *Two particles impinge on one another, and at the instant of impact become rigidly connected; to find their common velocity after impact and the magnitude and direction of the impulsive stress between them.*



Let the masses of the particles be m, m', and let oa and oa' represent their respective velocities before impact, and let g be the centroid of the velocity diagram, so that $m \,.\, ag = m' \,.\, ga'$. og represents the velocity of the centre of mass, which (§ 86) is unaltered by the impact. og therefore also represents the common velocity of the masses after impact.

The impulse at impact on the particle m is $m(\overline{og} - \overline{oa})$, or $m \,.\, \overline{ag}$, which $= \dfrac{mm'}{m+m'} \,.\, \overline{aa'}$, while similarly that on m' is $\dfrac{mm'}{m+m'} \,.\, \overline{a'a}$.

The direction of the impulse is therefore that of the relative velocity before impact; its magnitude is half the harmonic mean of the masses × the relative speed before impact.

Kinetic energy is always lost in this class of impact, as noted in § 114, Cor. 2. The amount lost is that relative to the centre of mass, or $\frac{1}{2}m \,.\, ga^2 + \frac{1}{2}m' \,.\, ga'^2$, which at once reduces to $\dfrac{1}{2}\dfrac{mm'}{m+m'}V^2$, where V is the relative speed before impact.

Example. Deduce this last result also from the expression given in § 111 for the work done by an impulse.

139. Direct Impact of Spherical Particles.

DEFINITION. *Two spherical particles are said to impinge* **directly** *when the velocity of each just before impact is parallel to the straight line joining their centres at the instant of impact.*

We shall assume as the basis of our theory of impact that in the direct impact of two spherical particles A and B the magnitude of the velocity of A relative to B after impact bears a constant ratio to its magnitude before impact, and that its direction is reversed.

The value of this ratio is denoted by e, and is called the coefficient of restitution; e is to be understood to be constant for particles of the same two materials however the masses and velocities are changed.

If $e=1$, the particles are said to have perfect restitution. No materials for which restitution is perfect exist in nature. For two glass spheres e is equal to $\frac{15}{16}$, or very nearly unity.

If $e<1$, as is always the case in nature, the particles are said to have imperfect restitution.

If $e=0$, the particles do not separate after impact and are said to be without restitution.

Until recently e was called the coefficient of elasticity, and particles were called perfectly elastic ($e=1$), imperfectly elastic ($e<1$), inelastic ($e=0$); the phrase *coefficient of elasticity* has, however, a definite and different meaning in the Theory of Elasticity, and is therefore better avoided.

The above rule was given by Newton as the result of experiments carried out by him; it is a strictly empirical result; the question of its place in the theory of elastic bodies is one of great difficulty, and is by no means settled. The rule has been repeatedly tested by experiment, and its accuracy for spherical particles seems established.

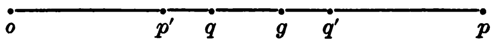
The time during which two impinging spheres are in contact is, according to Thomson and Tait, probably less than one-thousandth part of a second if the spheres do not exceed a yard in diameter, and are of such degrees of rigidity as copper, steel, or glass. The stress between the bodies at impact is therefore conveniently regarded as an *instantaneous impulse*; if the bodies are acted on by finite forces, their weights for instance, the changes of velocity at the instant of impact will be the same (§ 68) as if these forces did not exist.

140. *Two spherical particles whose coefficient of restitution is* e *impinge directly. To find their subsequent motion.*

Let m, m' be the masses of the two particles,

u, u' their velocities before impact,
v, v' their velocities after impact,

all the velocities being reckoned positive if their sense is from left to right; and let m be to the left of m' when the impact takes place, so that $u > u'$.



Denote the velocities u, u' by op, op',
and v, v' by oq, oq'.

Then (i.) the momentum, and therefore the velocity of the centre of mass, is unaltered by the impact.

Denote this velocity by og. g is the centroid of the velocity diagram, and hence

$$m \,.\, gp = m' \,.\, p'g \text{ and } m \,.\, qg = m' \,.\, gq'.$$

(ii.) The relative velocities before and after impact are, by Newton's experiments, in the ratio $1 : -e$. These relative velocities are pp', qq'.

$$\therefore\ qq' = -e \,.\, pp',$$

or, taking account of sense, $qq' = e \,.\, p'p$.

Now g divides qq' and pp' in the same ratio, and therefore

$$qg = e \,.\, gp, \quad q'g = e \,.\, gp',$$

or, *relative to the centre of mass, the velocity of each particle after impact* $= -$e (*its velocity before impact*)..............................(A)

$$\therefore\ qp = qg + gp = (1+e)gp$$
$$= (1+e)\frac{m'}{m+m'} \,.\, p'p$$
$$= (1+e)\frac{m'}{m+m'}(u-u').$$

Hence if μ be the magnitude of the stress between the balls,

$$\mu = m \,.\, qp$$
$$= (1+e)\frac{mm'}{m+m'}(u-u'),$$

or in words,

The impulse between the particles equals $(1+\text{e}) \times$ *half the harmonic mean of the masses* $\times$ *the relative velocity before impact*, being greater than the value it would have had if the particles had been without restitution in the ratio $1+e : 1$.

The change of velocity of m is $-\frac{\mu}{m}$,

that of m' is $+\frac{\mu}{m'}$.

Thus $$v = u - \frac{\mu}{m} = u - (1+e)\frac{m'}{m+m'}(u-u'),$$

$$v' = u' + \frac{\mu}{m'} = u' + (1+e)\frac{m}{m+m'}(u-u').$$

141. The problem can, of course, also be solved by means of the equations,

$$mv + m'v' = mu + m'u', \text{ ....(i.)}$$

expressing the constancy of the momentum, and

$$v' - v = -e(u'-u), \text{ ....(ii.)}$$

expressing that the relative velocity changes in the ratio $1 : -e$. But we have preferred to use the diagram of velocities as it gives considerable insight into the structure of the result.

The theorem proved incidentally at (A), viz.:

If two spherical particles whose masses are m, m′ *impinge directly, the velocity of each relative to the centre of mass is altered in the ratio* 1 : −e, is so important that we give an independent Cartesian proof of it.

The velocity of the centre of mass is $\frac{mu+m'u'}{m+m'}$.

Hence the velocity of m relative to this point before impact is

$$u - \frac{mu+m'u'}{m+m'} = \frac{m'}{m+m'}(u-u').$$

Similarly the velocity of m relative to this point after impact is $\frac{m'}{m+m'}(v-v')$, and the theorem follows since $v'-v = -e(u'-u)$.

Examples.

1. If the restitution is perfect and the masses equal, prove by inspecting the diagram of velocities that the particles on impinging interchange velocities.

2. Hence discuss the following: In a straight line on a smooth horizontal table lie n balls, each of mass m, followed by n balls, each of mass m'. A ball of mass m impinges directly on the first of those of mass m with velocity V parallel to the line of centres; examine the state of the system at any subsequent instant, all the balls having perfect restitution.

3. If a ball of mass m impinge directly on a ball of mass m' at rest, prove from the velocity diagram that m will be reduced to rest if $m=em'$.

4. A ball of mass 4 grams impinges directly on a ball of mass 3 grams, the former moving with a velocity 13 cms. a second, the latter with velocity 6 cms. a second. If the coefficient of restitution is $\frac{1}{3}$, find the velocities after impact, and the magnitude of the stress between the balls.

142. Let us study the expression for the magnitude of the impulse between the particles, viz., $(1+e)\frac{mm'}{m+m'}(u-u')$, a little more closely.

The change of velocity of the particle m, which is equal to

$$-(1+e)\frac{m'}{m+m'}(u-u'),$$

may be regarded as made up of the two parts

$$-\frac{m'}{m+m'}(u-u')=pg \text{ (fig. § 140)}$$

and
$$-e\,.\frac{m'}{m+m'}(u-u')=e\,.\,pg=gq.$$

The first part reduces the velocity to that of the centre of mass, the second further reduces it to the final velocity v.

Now we may divide the very small time of impact into two periods; during the first of these the spheres are compressing each other, during the second they are regaining their shapes, the alteration of form being of course very slight; let us call the first period that of compression, the second that of restitution; at the instant dividing the periods the spheres are moving with a common velocity, viz., that of the centre of mass. Let us call the impulses of the reaction between the spheres for these periods respectively I_1, I_2. Then it is clear that I_1 produces in the mass m the change of momentum $m\,.\,pg$, while I_2 produces the further change $m\,.\,gq=em\,.\,gp$. Hence

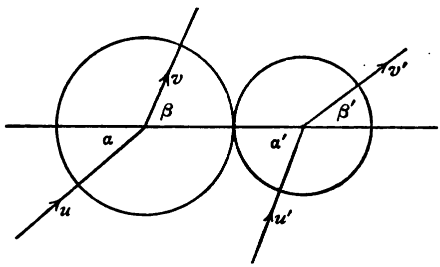
$$I_2=eI_1,$$

or the impulse of the reaction during restitution is equal to e times the impulse of the reaction during compression.

143. Oblique Impact of smooth spherical particles. The oblique impact of smooth spherical particles is usually treated by the assumption that Newton's experimental result for direct

impact applies *mutatis mutandis* when the impact is oblique, or that when two spherical particles impinge with any velocities, the resolved relative velocity parallel to the line of centres at the instant of impact is altered by the impact in the ratio $1 : -e$, e being a constant defined as before.

Two smooth spherical particles of masses m, m′ *impinge obliquely. To find the impulsive stress between them during the impact, and their velocities after impact.*



Let the velocities of the particles before impact be u, u' inclined to the line joining the centres at the instant of impact at angles a, a', and let the velocities after impact be v, v' inclined to the same line at angles β, β', the positive sense for resolved parts parallel to the line of centres being from left to right.

Then (i.) since the particles are smooth, there is no impulse on either particle perpendicular to the line of centres, and therefore the resolved velocities perpendicular to this line are unaltered, and therefore

$$\left.\begin{aligned} v\sin\beta &= u\sin a, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)\\ v'\sin\beta' &= u'\sin a'. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}\right\}$$

(ii.) Since there is no impulse on the *system* parallel to the line of centres, the momentum resolved in this direction is unaltered, or

$$mv\cos\beta + m'v'\cos\beta' = mu\cos a + m'u'\cos a'. \ldots\ldots\ldots(3)$$

(iii.) Newton's experimental result gives

$$v'\cos\beta' - v\cos\beta = -e(u'\cos a' - u\cos a). \ldots\ldots\ldots(4)$$

Equations (1), (2), (3), and (4) are sufficient to determine the unknown quantities v, v', β, β'.

Instead of solving these equations directly, it will be better to note that, writing $u\cos a$ for u, $u'\cos a'$ for u', etc., the

theorem of § 141 still holds; in other words, *the velocity of each particle relative to the centre of mass resolved parallel to the line of centres is altered by impact in the ratio* $1 : -e$.

Hence, since the velocity of the centre of mass remains unaltered by the impact, the complete change in the resolved velocity of m = the change in that velocity relative to the centre of mass

$$= -e \cdot \frac{m'}{m+m'}(u\cos\alpha - u'\cos\alpha') - \frac{m'}{m+m'}(u\cos\alpha - u'\cos\alpha')$$

$$= -(1+e)\frac{m'}{m+m'}(u\cos\alpha - u'\cos\alpha').$$

Hence, if μ be the value of the impulse between the particles,

$$\mu = (1+e)\frac{mm'}{m+m'}(u\cos\alpha - u'\cos\alpha').$$

The corresponding changes of velocity being $-\frac{\mu}{m}$ for the particle m, and $+\frac{\mu}{m'}$ for the particle m', we have thus

$$v\cos\beta = u\cos\alpha - \frac{\mu}{m}, \quad \text{.........................(5)}$$

$$v'\cos\beta' = u'\cos\alpha' + \frac{\mu}{m'}. \quad \text{.........................(6)}$$

Hence from (1) and (5), squaring and adding,

$$v^2 = u^2 - 2u\cos\alpha \cdot \frac{\mu}{m} + \frac{\mu^2}{m^2}, \quad \text{......................(7)}$$

$$v'^2 = u'^2 + 2u'\cos\alpha' \cdot \frac{\mu}{m'} + \frac{\mu^2}{m'^2}, \quad \text{......................(8)}$$

giving v and v' in terms of known quantities.

Lastly, dividing (5) by (1) and (6) by (2),

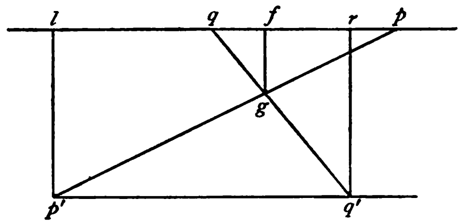
$$\left.\begin{aligned} \cot\beta &= \cot\alpha - \frac{\mu}{mu\sin\alpha}, \quad \text{.....................(9)} \\ \cot\beta' &= \cot\alpha' + \frac{\mu}{m'u'\sin\alpha'}, \quad \text{..................(10)} \end{aligned}\right\}$$

giving the directions of v and v'.

Corollary. It follows as in § 142, substituting resolved velocities for u, u', that if I_1, I_2 be the impulses during compression and restitution respectively, $I_2 = eI_1$.

144. The student will find it instructive to obtain equations (7), (8), (9), and (10) of the last paragraph directly from the velocity diagram, which we may construct as follows:

Let op, op' represent the velocities before, oq, oq' the velocities after impact. o is not indicated in the figure. Taking g, in pp', so that $m \, . \, gp = m' \, . \, p'g$, og is the velocity of the centre of mass, which remains unaltered. Hence qgq' is a straight line, and $m \, . \, qg = m' \, . \, gq'$.



pq, $p'q'$ being the changes of velocities are parallel to the line of centres.

If l, r be the respective projections of p', q' on pq or pq produced, pl, qr are the resolved relative velocities parallel to the line of centres before and after impact. Hence, by Newton's experiment, $qr = e \, . \, lp$, and if f be the projection of g on pq, the student will easily prove by similar triangles that $qf = e \, . \, fp$, $fr = e \, . \, lf$. The equations now easily follow.

145. Certain general results may be here remarked.

(1) In the oblique impact of two spherical particles if the initial velocities are increased in any the same ratio without altering their directions, the final velocities are in the same directions as before, and are also increased in this ratio; moreover, the magnitude of the stress is increased in this ratio.

The proof merely consists in remarking that the standard figure of § 144 will, if magnified until op and op' are increased in the required ratio, represent equally well the velocity diagram of the new motion.

(2) *If two spherical particles whose restitution is perfect impinge obliquely, their relative velocities before and after impact are equally inclined to the line of centres.*

This is evident, for pp', qq' are the relative velocities, and in this case $qf = fp$.

(3) The conditions that the particles should interchange velocities are :

(i.) They must have the same resolved velocity perpendicular to the line of centres.

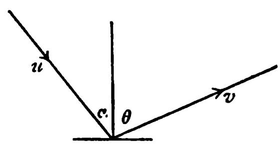
(ii.) Their masses must be equal.

(iii.) Their restitution must be perfect.

The reader will easily deduce these conditions from an inspection of the diagram, making p and q' coincide, as also p' and q.

146. Impact of a Particle on a Smooth Fixed Plane. If e be the coefficient of restitution for the materials of which the plane and the particle are composed, the resolved velocity of the particle perpendicular to the plane is altered in the ratio $1 : -e$, while the resolved velocity parallel to the plane is unaltered.

Hence if u be the velocity of the particle before impact, v that after impact, and if u, v be inclined to the normal to the plane



at angles α, θ respectively, and e be the coefficient of restitution between the particle and the plane,

$$v \sin\theta = u \sin\alpha, \quad v\cos\theta = -eu\cos\alpha,$$

whence $\quad v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$ and $\cot\theta = -e\cot\alpha$,

the negative sign showing that the directions of u and v lie on opposite sides of the normal. If m be the mass of the particle, the impulse on it is clearly $mu\cos\alpha(1+e)$.

147. If the plane is rough, the coefficient of friction being μ, there is at each instant of the small time during which the contact lasts a friction equal to μ times the normal pressure; hence the *impulse* of the friction $=\mu$ times the normal impulse

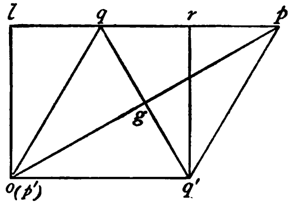
$$= \mu \,.\, mu\cos\alpha(1+e).$$

Hence the resolved velocity of the particle parallel to the plane after impact is $u\sin\alpha - \mu u\cos\alpha(1+e)$, and θ the angle the direction of motion after impact makes with the normal is given by

$$\cot\theta = -\frac{e\cos\alpha}{\sin\alpha - \mu\cos\alpha(1+e)}.$$

Examples.

1. There are two smooth spheres of equal mass, one at rest, and their directions of motion after impact are both inclined at 30° to the direction of motion of the impinging sphere; prove that the coefficient of restitution $=\frac{1}{3}$.

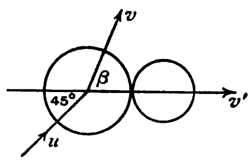


Drawing the velocity diagram, with the notation of § 144, p' coincides with o, g is the middle point of pp', and $qoq'p$ is a rhombus, angle qoq' being 60°.

Hence $lq=qr=rp$; but $qr=e\,.\,lp$;

$$\therefore\ e=\tfrac{1}{3}.$$

2. One smooth ball impinges on another, which is at rest; if the restitution is perfect and the original direction of the motion of the striking ball be inclined at an angle of 45° to the line joining the centres of the balls, find the angle between the directions of its motion before and after impact.



Here, with the usual notation, $u'=0$, $\alpha=45°$, $\beta'=0$, since the second ball moves off parallel to the line of centres.

For equation (1) we have $v\sin\beta=u\sin 45°$.

Equation (2) is identically satisfied.

For equation (3) we have $mv\cos\beta+m'v'=mu\cos 45°$.

And for equation (4), $v'-v\cos\beta=+u\cos 45°$.

From (3) and (4),

$$mv\cos\beta+m'\left(v\cos\beta+\frac{u}{\sqrt{2}}\right)=\frac{mu}{\sqrt{2}},$$

or

$$(m+m')v\cos\beta=(m-m')\frac{u}{\sqrt{2}}.$$

Also, from (1), $$v \sin\beta = \frac{u}{\sqrt{2}},$$

$$\therefore \quad \tan\beta = \frac{m+m'}{m-m'};$$

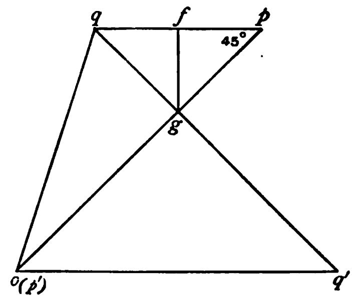
$\therefore$ angle between directions of motion of the ball whose mass is m

$$= 225^\circ - \beta = \tan^{-1}\frac{1-\dfrac{m+m'}{m-m'}}{1+\dfrac{m+m'}{m-m'}}$$

$$= \tan^{-1}\left(-\frac{m'}{m}\right),$$

or the acute angle $$= \tan^{-1}\frac{m'}{m}.$$

Or, by the diagram, thus;



Here op' is zero, so that p' and o coincide. Also $qf = fp$, since $e = 1$.

Also, since angle $opq = 45^\circ$, $gf = fp$,

$$\therefore \quad \widehat{qgp} \text{ is a right angle, and } qg = gp;$$

$$\therefore \quad \tan\widehat{poq} = \frac{qg}{og} = \frac{gp}{og} = \frac{m'}{m}.$$

3. A smooth sphere B at rest is struck by another smooth sphere A, so that, whatever the direction of A's motion before collision may be, A and B after collision move at right angles; find the value of the coefficient of restitution.

4. A smooth ball strikes another which is at rest, the angle between the line joining the centres of the balls and the direction

of motion of the first ball being α. This ball moves off in a direction perpendicular to its original direction. Show that the masses of the balls are in the ratio $e\cos^2\alpha - \sin^2\alpha : 1$, where e is the coefficient of restitution.

5. Two equal balls moving in parallel but opposite directions impinge obliquely. If after impact their lines of motion are perpendicular, prove that the sum of the squares of their speeds after impact is equal to the square of the difference of their speeds before impact. Also find the angle which their direction of motion before impact makes with the line of impact that this may be the case.

The diagram gives a compact solution.

6. Six equal spheres are placed with their centres at the angular points of a regular hexagon, and another equal sphere lies within the hexagon in contact with one of them so that the common tangent passes through the centre of the next, and is projected towards the next parallel to this tangent; the coefficient of restitution of all the spheres being unity, show that the projected sphere will impinge on them all, and will after the last impact be moving in its original direction. Find its speed after the last impact, and the speeds and directions of motion of the others.

7. A ball A is dropped from a height h, and subsequently an equal ball is projected vertically upwards with a speed due to the height h, and meets the first ball in direct impact. If the restitution is perfect, discuss the subsequent motion.

8. A smooth ball impinges on another ball at rest; prove that when the deviation of the path of the striking ball after impact from its original direction is a maximum

(i.) the initial velocity of the striking ball makes the same angle with the line of centres as the final velocity makes with a perpendicular to this line;

(ii.) this angle $= \tan^{-1}\dfrac{v}{u} = \tan^{-1}\sqrt{\dfrac{m-em'}{m+m'}}$,

u and v being the initial and final speeds of the striking ball, m, m' the masses of the balls, e the coefficient of restitution;

(iii.) the angle of maximum deviation is

$$\tan^{-1}\frac{(1+e)m'}{2\sqrt{(m-em')(m+m')}}.$$

What is the maximum deviation when $m < em'$?

[These results are easily obtained by geometry from the velocity diagram, noting that p' coincides with o, that the ratio $pq : pl$ is constant, and that op may be taken, without loss of generality, to be of constant length (§ 145).]

9. Prove that in order to produce the greatest deviation in the direction of a smooth billiard-ball of diameter a, by impact on another equal ball at rest, the former must be projected in a direction making an angle $\sin^{-1}\frac{a}{c}\sqrt{\frac{1-e}{3-e}}$ with the line (of length c) joining their centres, e being the coefficient of restitution.

10. A smooth sphere impinges against another of equal mass, the restitution being perfect. Prove (from the diagram, or otherwise) that the impinging sphere always moves off at right angles to the line of centres. What happens when the impact is direct?

11. A smooth ball of given mass moving with given *speed* comes into collision with another ball of given mass moving with given *velocity*. Give a geometrical construction for the maximum deviation of the latter, the line of centres remaining the same in all cases.

148. Loss of Kinetic Energy by Impact. When two smooth spheres whose restitution is imperfect impinge, a certain amount of kinetic energy disappears, and is converted into heat.

We proceed to estimate the kinetic energy which thus changes shape. Now

(1) The kinetic energy of the centre of mass is unaltered by the impact. The whole loss of energy must be therefore from the kinetic energy relative to the centre of mass.

(2) Resolve the velocities of the spheres parallel and perpendicular to the line of centres at impact. That part of the kinetic energy which depends on the perpendicular components remains unaltered. Hence we have only to study the kinetic energy relative to the centre of mass due to the components parallel to the line of centres.

Let V, V' be the relative velocities of the particles, resolved parallel to the line of centres, before and after impact, both estimated in the same sense (one of them will of course be negative).

The resolved velocities of m relative to the centre of mass are thus

$$-\frac{m'}{m+m'}V \text{ and } \frac{m'}{m+m'}V',$$

while those of m' are $\quad \frac{m}{m+m'}V$ and $-\frac{m}{m+m'}V'$.

Hence the loss of kinetic energy is

$$\frac{1}{2}m\cdot\left(\frac{m'}{m+m'}V\right)^2-\frac{1}{2}m\cdot\left(\frac{m'}{m+m'}V'\right)^2$$
$$+\frac{1}{2}m'\left(\frac{m}{m+m'}V\right)^2-\frac{1}{2}m'\left(\frac{m}{m+m'}V'\right)^2$$
$$=\frac{1}{2}\frac{mm'}{m+m'}(V^2-V'^2)$$
$$=\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)V^2,\text{ since } V'=-eV$$
$$=\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)(u\cos\alpha-u'\cos\alpha')^2,$$

where u, u', α, α' have their usual meanings.

The loss is thus always positive, vanishing, however, when $e=1$.

Examples.

1. Deduce this result:

(*a*) From the formula of § 111 for the work done by an impulse.
(*b*) From the velocity diagram.

2. Two smooth balls impinge obliquely on one another, and at the instant of impact they fire a small quantity of detonating powder, which by its explosion imparts to each of them (entirely in the line of centres) a momentum bearing to the momentum transferred by ordinary impact the ratio $e' : 1+e$. Prove that the increase of kinetic energy due to impact is

$$\tfrac{1}{2}(\overline{e+e'}^2-1)\frac{mm'}{m+m'}(u-u')^2,$$

where m, m' are the masses of the two balls, u, u' their velocities resolved parallel to the line of centres before impact.

3. Two smooth spheres, whose restitution is perfect, impinge directly. Show that the energy exchanged is the product of the speed of their centre of mass, of the relative speed before impact, and of the harmonic mean of their masses.

4. One smooth sphere impinges on another at rest; if half the kinetic energy is lost, prove that the coefficient of restitution must be less than $\dfrac{1}{\sqrt{2}}$.

5. Prove that the energy lost in the collision of two smooth spheres is

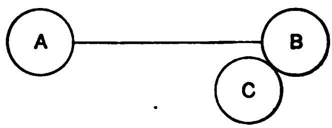
$$\frac{1}{2}\cdot\frac{1-e}{1+e}\left(\frac{1}{m}+\frac{1}{m'}\right)\cdot I^2,$$

where I is the momentum transferred from ball to ball.

149. When a spherical particle impinges simultaneously on two others, or when the impinging particles are in any way *constrained*, it is clear that Newton's rule as to the relative velocity after impact is no longer directly applicable. In this case it is usual to assume, for each pair of bodies which impinge, that the impulse during restitution is equal to e times the impulse during compression (see § 142). A single example will suffice to make the method of procedure clear.

Example. *Two equal balls on a smooth plane are connected by a tight inextensible string; a third equal ball impinges on one of them directly with speed* V *in a line making an angle* α *with the string, so that the string is jerked; prove that the initial speed of the ball that is struck is*

$$\frac{V(1+e)\sqrt{1+3\sin^2\alpha}}{3+\sin^2\alpha}.$$



Denote the balls connected by the string by A and B, and let the third ball C strike B. Without loss of generality we may take the masses of each of the three balls to be unity.

Then, at the end of the period of compression, the balls B and C will be moving with a common velocity parallel to the line of their centres; the velocity of A will be, both at the end of this period and at the instant when B and C separate, along the string.

Let then

I_1 be the impulse between B and C during compression,

I_2 " " " restitution;

u, u' the velocities of B respectively parallel and perpendicular to the string at the end of the period of compression.

v, v' the final velocities of B in these directions.

Then the corresponding velocities of A are u, v along the string.

Now the velocity of C is reduced by the impulse I_1 from V to

$$u\cos\alpha + u'\sin\alpha,$$

this latter being the velocity of B, at the end of the period of compression, resolved in the direction CB.

$$\therefore \quad I_1 = V - (u\cos\alpha + u'\sin\alpha). \qquad \text{(i.)}$$

And we further have, resolving parallel and perpendicular to the string for the momentum of the system (A, B),

$$I_1 \cos a = 2u, \quad \text{(ii.)}$$

$$I_1 \sin a = u', \quad \text{(iii.)}$$

$$I_2 \cos a = 2v - 2u, \quad \text{(iv.)}$$

$$I_2 \sin a = v' - u'; \quad \text{(v.)}$$

and finally, $$I_2 = e \,.\, I_1. \quad \text{(vi.)}$$

From (ii.), (iv.), and (vi.) we obtain

$$v = (1+e)u,$$

and from (iii.), (v.), and (vi.), $v' = (1+e)u'$.

Also, by substituting for u and u' in (i.) their values found from (ii.) and (iii.), we have

$$I_1 = V - I_1 \frac{\cos^2 a}{2} - I_1 \sin^2 a,$$

giving $$I_1 = \frac{2V}{3+\sin^2 a},$$

whence $$u = \frac{V \cos a}{3+\sin^2 a}, \qquad u' = \frac{2V \sin a}{3+\sin^2 a},$$

$$v = \frac{(1+e)V \cos a}{3+\sin^2 a}, \quad v' = \frac{(1+e)\,.\,2V \sin a}{3+\sin^2 a},$$

and the final speed of B

$$= \sqrt{v^2 + v'^2} = \frac{(1+e)V\sqrt{1+3\sin^2 a}}{3+\sin^2 a}.$$

Had it been desired to find the final speed of the ball C, we should have had, denoting this speed by V',

$$I_1 + I_2 = V - V',$$

or $$I_1(1+e) = V - V',$$

whence V' is at once found.

150. Continuous Impact. We have remarked (§ 68) that the effect of a constant succession of impulses on a particle, when the number of impulses is indefinitely increased, and their magnitude and the intervals between them indefinitely diminished, approximates to that of a force. The method of dealing with such cases is illustrated in the following example.

A child's toy consists of a metal tube $\frac{1}{8}$ in. in diameter, bent towards one end at right angles, and terminating in a cup. In the cup rests a ball of $1\frac{1}{2}$ in. diameter, consisting of a very light envelope filled with air; the mass of the envelope and the air it contains are equal to 24 times that of an equal volume of air. On blowing through the tube, a column of air whose diameter is equal to that of the tube impinges on the ball, raises it, and supports it steadily. Find the speed of the current of air in feet per second, assuming that its speed after striking the ball is negligible.

Since the column of air delivered is of small section, we may assume that all the impulses of the particles of air on the ball are vertical.

Now, v being the speed of the current of air in feet per second, the volume of air delivered in a small interval τ

$$=\pi\left(\frac{1}{16\times 12}\right)^2 . v\tau \text{ cubic feet.}$$

Its mass therefore is

$$d . \pi\left(\frac{1}{16\times 12}\right)^2 . v\tau,$$

where d is the mass of 1 cubic foot of air.

Hence the momentum destroyed by impact in the interval τ is

$$d . \pi . \left(\frac{1}{16 . 12}\right)^2 . v^2\tau,$$

and the momentum destroyed per second is

$$d . \pi\left(\frac{1}{16 . 12}\right)^2 . v^2.$$

This is equivalent to the upward thrust on the ball.

Hence, if M be the mass of the ball, we have

$$d . \pi\left(\frac{1}{16 . 12}\right)^2 . v^2 = Mg,$$

or

$$v^2 = \frac{M}{d}\frac{(16 . 12)^2}{\pi} . g,$$

Now the volume of the ball

$$=\frac{4}{3}\pi\left(\frac{3}{4\times 12}\right)^3 \text{ c. feet}=\frac{\pi}{16\times 12\times 16}\text{ c. feet.}$$

$$\therefore\quad M:\frac{\pi d}{16^2 . 12}::24:1,$$

or

$$\frac{M}{d}=\frac{2\pi}{16^2}.$$

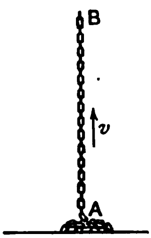
$$\therefore\quad v^2=\frac{2\pi}{16^2}\cdot\frac{(16\times 12)^2}{\pi}\,.\,32,\text{ putting } g=32,$$

$$=64\times 12^2.$$

$$\therefore\quad v=96 \text{ feet per second,}$$

or rather more than the speed of an express train.

151. Motion of Heavy Chains. A heavy string or chain may be considered (§ 70) as the limit of a line of particles connected by light strings, when the number of particles is indefinitely increased, and the intervals between them indefinitely diminished. The student will find no difficulty in writing down equations for the motion of such a chain from this point of view. The procedure we shall adopt will differ slightly from this.



Consider a heavy chain whose mass *per unit length* is m, partially coiled up on a horizontal table at A, while the straight part AB is moving vertically upwards with speed v. As each link (or particle) is picked up and started into motion with speed v there will be a small impulse on the chain at A, and the constant succession of these impulses will be equivalent to a finite tension at A. In fact, in a small time τ a length $v\tau$ is started into motion; the mass of this length is $mv\tau$, and its momentum changes from zero to $mv\tau . v$ or $mv^2\tau$; consequently the value of the tension at $A=\dfrac{mv^2\tau}{\tau}=mv^2$.

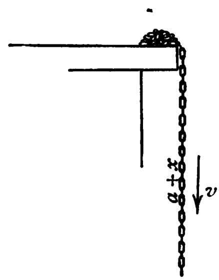
Similarly, if the chain is *descending*, the pressure on the table will be equal to the weight of the part coiled up increased by the force requisite to destroy the momentum of the links as they arrive at A; this latter force is, as before, mv^2.

When at some part of the chain the links are thus being one by one jerked into motion, or brought suddenly to rest, the

equation of energy cannot be directly applied; for we have learnt that energy is always lost from the system in inelastic impact. A simple example will bring this clearly home to the student. Suppose that a chain of length l and mass per unit length m is coiled loosely in the hand, and that the coil is let fall while one end is held fast. When the chain is hanging straight, the work that has been done by the weights of the various links is equal to $m\,.\,l\,.\,g\,.\,\frac{l}{2}$, or $\frac{1}{2}ml^2g$, but the chain has no kinetic energy; consequently an amount of energy equal to $\frac{1}{2}ml^2g$ has disappeared from the system.

It is easy to find an expression for the rate at which energy is leaving the system at the point A (above). For a mass $mv\,.\,\tau$ is started in time τ with speed v. Hence (expression for the work done by an impulse, § 111) the energy lost in time τ is $mv^2\,.\,\tau\,.\,\frac{v+0}{2}$, or $\frac{1}{2}mv^3\tau$. Consequently energy is leaving the system at a rate $\frac{1}{2}mv^3$ per second.

Example 1. *A chain whose mass per unit length is* m *is coiled up near the edge of a table, with a length* a *hanging over the edge. The system starts from rest. Find the speed when an additional length* x *has passed over the edge.*



Let v be the speed, f the acceleration when the additional length x has passed over. The mass of the vertical part of the chain is thus $m(a+x)$.

The external forces are its weight $mg(a+x)$ and the tension at the top, which is equal to mv^2.

Hence, by the Second Law,

$$m(a+x)f = mg(a+x) - mv^2,$$

or $$v^2 + (a+x)f = (a+x)g. \quad \text{............(i.)}$$

Now divide x into n intervals, each equal to y, so that $x = ny$, and let $v_0, v_1, v_2, \ldots v_n$ be the speeds when lengths $0, y, 2y, 3y, \ldots$ of chain have passed over the edge, then, provided we diminish n indefinitely, in accordance with the kinematical principles laid down in Chap. II. we may write for f in the above equation

$$f = \frac{v_n^2 - v_{n-1}^2}{2y},$$

whence, substituting for f in (i.) and writing v_n^2 for v^2,

$$v_n^2 \cdot 2y + (a+ny)(v_n^2 - v_{n-1}^2) = (a+ny)2gy.$$

$$\therefore \quad v_n^2(a+\overline{n+2}y) - v_{n-1}^2(a+ny) = (a+ny)2gy,$$

or, multiplying through by $a+\overline{n+1}y$,

$$v_n^2(a+\overline{n+2}y)(a+\overline{n+1}y) - v_{n-1}^2(a+\overline{n+1}y)(a+ny)$$
$$= 2gy(a+ny)(a+\overline{n+1}y).$$

Now $v_0 = 0$. Hence, by ordinary algebra (see Charles Smith's Algebra, Chap. xxv., § 318, *Summation of Series*),

$$v_n^2(a+\overline{n+2}y)(a+\overline{n+1}y)$$
$$= 2gy \cdot \frac{(a+ny)(a+\overline{n+1}y)(a+\overline{n+2}y) - a(a+y)(a+2y)}{3y},$$

whence, diminishing y indefinitely, and putting $ny = x$,

$$v^2(a+x)^2 = \frac{2g}{3}(\overline{a+x}^3 - a^3),$$

or

$$v^2 = \frac{2g}{3}\left[a + x - \frac{a^3}{(a+x)^2}\right].$$

Corollary. If $a = 0$, $v^2 = \frac{2gx}{3}$, and the acceleration is constant and $= \frac{g}{3}$.

Example 2. *A uniform chain, a length* l *of which has mass* m, *lies coiled completely up on a horizontal table. To one end is attached a light inextensible string which passes over a smooth pulley vertically above the coil; to the other end of the string is attached a mass* m; *the system being initially at rest, discuss the motion.*

Let v be the speed, f the acceleration of the system when a length x of the chain is raised, and before the system first comes to rest.

At the end of the uncoiled part of the chain there is a tension $\frac{m}{l}v^2$.

We may regard the part of the system which is in motion at this instant as a mass $m + m \cdot \frac{x}{l}$ under the external forces $\left(m - m\frac{x}{l}\right)g$, and $\frac{m}{l}v^2$, in opposite senses (§ 84).

$$\therefore \quad \left(m+m\frac{x}{l}\right)f=\left(m-m\frac{x}{l}\right)g-\frac{m}{l}\cdot v^2.$$

$$\therefore \quad v^2+(l+x)f=(l-x)g.$$

As before, put $x=ny$, $f=\dfrac{v_n^{\,2}-v_{n-1}^{\,2}}{2y}$, $v=v_n$,

$$\therefore \quad v_n^{\,2}+(l+ny)\frac{v_n^{\,2}-v_{n-1}^{\,2}}{2y}=(l-ny)g,$$

whence $\qquad v_n^{\,2}(l+\overline{n+2}y)-v_{n-1}^{\,2}(l+ny)=2gy(l-ny).$

Multiplying by $l+\overline{n+1}y$,

$$\begin{aligned}v_n^{\,2}(l+\overline{n+2}y)(l+\overline{n+1}y)&-v_{n-1}^{\,2}(l+\overline{n+1}y)(l+ny)\\&=2gy(l-ny)(l+\overline{n+1}y)\\&=2gy[2l(l+\overline{n+1}y)-(l+ny)(l+\overline{n+1}y)];\end{aligned}$$

whence, since $v_0=0$, we have, by summation of series,

$$\begin{aligned}v_n^{\,2}(l+\overline{n+2}y)&(l+\overline{n+1}y)\\&=2gy\left[2l\frac{(l+\overline{n+1}y)(l+\overline{n+2}y)-(l+y)(l+2y)}{2y}\right.\\&\qquad\left.-\frac{(l+ny)(l+\overline{n+1}y)(l+\overline{n+2}y)-l(l+y)(l+2y)}{3y}\right];\end{aligned}$$

whence, increasing n indefinitely,

$$\begin{aligned}v^2(l+x)^2&=2g\left[l(\overline{l+x}^2-l^2)-\frac{(l+x)^3-l^3}{3}\right]\\&=2g\left[l^2x-\frac{x^3}{3}\right].\end{aligned}$$

$$\therefore \quad v^2=\frac{2g}{3}\cdot\frac{3l^2x-x^3}{(l+x)^2}.$$

The system next comes to rest when $v=0$, or $x=l\sqrt{3}$.

After this the chain again coils up; there is now, however, no tension at its lower extremity, but only a corresponding pressure on the table, which will not appear in the equation of motion of the chain. If f be the (downward) acceleration of the chain when a length x remains uncoiled, the new equation of motion is

$$(x+l)f=(x-l)g,$$

an equation by means of which v can also be found by elementary methods, but not so conveniently.

Since, however, energy is continually leaving the system, owing to the succession of impulses, at a rate $\frac{m}{l}v^3$, which is in general finite, it is evident that the chain after oscillating will at length come to rest with a length l uncoiled.

Examples on Chapter VI.

1. Two smooth balls impinge obliquely, so that their directions of motion are interchanged; prove that the product of their speeds is unaltered, and that, if their masses are equal, their directions of motion must be equally inclined to the line of impact, and their velocities must be interchanged. In this case, what is the coefficient of restitution?

2. Three equal balls A, B, C, whose coefficient of restitution is e, are at rest in a straight line on a smooth horizontal plane. If A be projected towards B, show that there will be (1) at least three impacts, (2) at least four if $e < 3 - 2\sqrt{2}$, and (3) at least five if $e + e^{-1} > 4 + 2\sqrt{5}$.

3. Two spheres, without restitution, of masses m and m' respectively are in contact, and m receives a blow through its centre in a direction making an angle a with the line of centres. Show that the kinetic energy generated is less than if m had been free in the ratio $m + m' \sin^2 a : m + m'$.

4. A number of particles originally in a straight line fall from rest and rebound from a horizontal plane whose restitution is imperfect. Prove that at any time the particles which have rebounded once will lie in a parabola, finding an equation for the curve in cartesian coordinates.

5. If a ball, whose restitution is imperfect, return to the same point after three reflexions on the inside of a smooth circle, the motion taking place on a smooth horizontal plane, one of the diagonals of the quadrilateral so described is a diameter of the circle.

6. Two unequal smooth spherical bodies of masses M and m are at rest at a distance a from one another, and are continually urged towards one another with the same force F of constant magnitude. Prove that they will always impinge on one another at the same point and will come to rest after a time

$$\frac{1+e}{1-e}\sqrt{\frac{2a}{F} \cdot \frac{mM}{M+m}}.$$

7. A ball, whose restitution is perfect, is projected vertically with velocity v_1 from a point in a rigid horizontal plane, and when

its velocity is v_2 an equal ball is projected vertically from the same point also with velocity v_1; show (i.) that the time that elapses between successive impacts of the two balls is $\frac{v_1}{g}$, (ii.) that the heights at which they take place are alternately

$$\frac{(3v_1 - v_2)(v_1 + v_2)}{8g} \text{ and } \frac{(3v_1 + v_2)(v_1 - v_2)}{8g},$$

(iii.) that the velocities of the balls at the impacts are equal and opposite and alternately $\frac{1}{2}(v_1 - v_2)$ and $\frac{1}{2}(v_1 + v_2)$.

8. Show that impact on a fixed obstacle is attended with greater loss of energy than impact on a body of the same material at rest but free to move; but that the kinetic energy of the striking body will be less after impact in the second case if its mass be less than $\frac{2e}{1-e}$ of the mass of the second body.

9. Two equal particles, whose coefficient of restitution is $\frac{1}{2}$, are hung from the same point by equal strings. They are drawn aside till the strings are in the same horizontal line, and are let go simultaneously. Find how many impacts must take place that the rebound of a particle may be reduced to less than 5 minutes of arc.

10. A particle is projected inside a straight smooth tube of mass equal to that of the particle closed at both ends and at rest on a smooth horizontal table. Prove that the distance travelled through by the tube when the particle has just made $(n+1)$ impacts is

$$\frac{a(1-e^n)}{e^n(1-e)} \text{ or } \frac{a(1-e^{n+1})}{e^n(1-e)},$$

according as n is even or odd, where $2a$ is the length of the tube, and e is the coefficient of restitution.

11. A smooth particle of mass m is at rest in an empty rectangular box of mass M which is free to move down a smooth plane, inclined at an angle α to the horizon, the lowest edge of the box being horizontal, and the particle being at its middle point. Suddenly the box is started down the plane with velocity V. Prove that, if the coefficient of restitution be unity, the particle will strike the top and bottom of the box after equal successive intervals of time; and that the spaces travelled by the box in the first and second of these intervals are as

$$V^2 + gl \sin \alpha : \frac{M-m}{M+m} V^2 + 3gl \sin \alpha,$$

where $2l$ is the length of the box.

12. There are two smooth balls, masses m_1, m_2; both are free to move inside a smooth circular tube, fixed in a horizontal plane;

prove that after n impacts the kinetic energy of the system is

$$\tfrac{1}{2}m_1u^2\frac{m_1+m_2e^{2n}}{m_1+m_2};$$

u being the initial speed of the ball m_1, the ball m_2 being initially at rest, and e being the coefficient of restitution.

13. A railway carriage of mass M moving with velocity v impinges on a carriage of mass M' at rest. The force necessary to compress a buffer through the full extent l is equal to the weight of a mass m. Assuming that the compression is proportional to the force, prove that the buffer will not be completely compressed if

$$v^2<2mgl\left(\frac{1}{M}+\frac{1}{M'}\right).$$

Prove also that if v exceeds this limit, and the backing against which the buffers are driven is without restitution, the ratio of the final speeds of the carriages is

$$Mv-\left\{2mM'gl\left(1+\frac{M'}{M}\right)\right\}^{\frac{1}{2}}:Mv+\left\{2mMgl\left(1+\frac{M}{M'}\right)\right\}^{\frac{1}{2}}.$$

14. Two equal small spherical beads are strung on a smooth horizontal circular wire, and projected in the same direction along the wire with speeds $5u$ and u respectively. Show that, if $e=\frac{1}{2}$, the impacts will all take place at the ends of the same diameter, and that the angle described by the first bead in the interval between the first and $n+1^{\text{th}}$ impact is

$$3\pi(2^n-1)-\frac{\pi}{2}\{1-(-1)^n\}.$$

15. Give a geometrical construction for the path of a ball which after impinging on the four sides of a billiard table returns to the place of projection.

16. A particle is projected along a smooth horizontal table so as to strike first one and then another of two perpendicular smooth boundaries whose coefficients of restitution are respectively e and e'. Show that, in order to make the deviation of the direction of the final path from the reversed direction of the original path as great as possible, the latter must make an angle

$$\tan^{-1}\{(\sqrt{e'}\sim\sqrt{e})/(\sqrt{e'}+\sqrt{e})\}$$

with the bisector of the angle between the boundaries, and that then the deviation is $\tan^{-1}\dfrac{1}{2}\left\{\sqrt{\dfrac{e'}{e}}\sim\sqrt{\dfrac{e}{e'}}\right\}$.

17. A wedge of mass M and angle α can move freely on a smooth horizontal plane; a smooth sphere of mass m strikes it in a direction perpendicular to its inclined face and rebounds. Prove that the

ratio of the speeds of the sphere just before and just after impact is $\dfrac{M+m\sin^2\alpha}{eM-m\sin^2\alpha}$, where e is the coefficient of restitution.

18. An inclined plane of mass M is capable of moving freely on a smooth horizontal plane; a smooth sphere of mass m is dropped vertically on its inclined face and rebounds. Show that the loss of kinetic energy is

$$\frac{1}{2}\cdot\frac{Mmu^2(1-e^2)\cos^2\alpha}{M+m\sin^2\alpha},$$

where e is the coefficient of restitution of the sphere, u its speed on reaching the plane, and α the inclination of the plane to the horizon.

19. A weightless rod is pivoted at its middle point, and is free to move in a horizontal plane. It has two small spheres, each of mass m, attached to its extremities, and another equal sphere moving in the horizontal plane strikes one of the former directly in a direction making an acute angle α with the rod; show that it will be reduced to rest by the impact if $\sin\alpha=\sqrt{2e}$, where e is the coefficient of restitution.

20. A smooth sphere of radius r and mass m is hung by a string above a horizontal table, and another smooth sphere of radius r' and mass m' is moving on the table; prove that the cotangent of the angle through which the direction of motion of the second sphere is deflected by a collision is

$$\frac{1}{mb}\{m'(r+r')^2+mb^2\}\{(r+r')^2-a^2-b^2\}^{-\frac{1}{2}};$$

the spheres being without restitution, and a and b being the vertical and horizontal distances of the centre of the first sphere from the path of the centre of the second before impact.

21. Two equal smooth spheres of radius r move with the same speed in opposite directions in parallel lines at a distance c apart; prove that the motion of each deviates on impact through a right angle if $c^2(1+e)=4er^2$, where e is the coefficient of restitution.

22. A small smooth billiard ball is at the centre of a rectangular billiard table whose sides are a and b. Another equal ball is placed at such a point of one of the sides a that when projected in the proper direction it will strike the other ball and drive it into one of the pockets at the corners, whilst it goes itself into the pocket in the middle of the opposite side a. Prove that the second ball must be placed at a distance from the centre of the side which it touches equal to $\frac{1}{2}ab^2\dfrac{1+e}{(1-e)a^2+2b^2}$, where e is the coefficient of restitution.

23. A smooth billiard ball of radius a is at rest with its centre at a distance b from a cushion. An equal ball is projected in a direction

making an angle θ with the cushion, and strikes the first ball so that the line of impact is perpendicular to the cushion. Show that, if θ be greater than $\cos^{-1}\frac{a}{b}$, the balls will kiss. (Assume that the balls and the cushion have perfect restitution.)

24. Three equal smooth billiard balls, whose size may be neglected, are placed at the points A, B, C. Show that it is not possible for A to cannon off B on to C, unless the angle ABC is greater than the obtuse angle whose sine is $\frac{1+e}{3-e}$, where e is the coefficient of restitution.

25. Two equal smooth balls lie in contact on a smooth table. A third equal ball impinges on them, its centre moving along a line nearly coinciding with the horizontal common tangent at the point of contact. Assuming that the periods of the two impacts do not overlap, prove that the ratio of the speeds which either ball will receive according as it is struck first or second, is $4 : 3 - e$, where e is the coefficient of restitution.

26. Two equal smooth spheres, each of mass m, are in contact on a smooth horizontal table, and a third equal sphere of mass m' impinges symmetrically on them. Show that this sphere is reduced to rest by the impact if $2m' = 3em$, e being the coefficient of restitution, and find the loss of kinetic energy by the impact.

27. Two smooth equal spheres, of radius a, rest in contact on a smooth horizontal plane. A smooth sphere, of radius b, falling with its centre vertically above the point of contact, strikes the spheres. Prove that the speed of the falling sphere is altered by the impact in the ratio $b^2 : 2a(2a+b)+b^2$, where the density of the spheres is the same, and the spheres are without restitution.

28. A weight of mass m, and a bucket of mass m', are connected by a light inextensible string which passes over a smooth pulley. These bodies are released from rest when a particle whose mass is p and coefficient of restitution e falls with vertical velocity V upon the bucket. Prove that a second collision will occur between the particle and the bucket after time $e(m+m')V/mg$, and find the condition that the bodies may then be in their initial positions.

29. Two equal scale pans, each of mass m, hang at rest over a smooth pulley. A particle of mass M without restitution is dropped from a height h into one scale pan, and at the same instant a particle of equal mass, but whose coefficient of restitution is e, is dropped from an equal height into the other scale pan. Prove that every impact occurs when the pans are in their original positions, and that the total space described by either pan before the motion ceases is

$$\frac{e^2}{1-e^2} \cdot \frac{M(2m+M)}{2(m+M)^2} \cdot h.$$

30. One end of a uniform chain is held above a fixed horizontal plane without restitution; part of the chain hangs vertically, and the rest is coiled up on the plane; the upper end of the chain is then released and the vertical part falls. Show that at any time the pressure on the plane is less than three times the weight of chain lying on the plane at that time, by twice the weight of chain originally lying on the plane.

31. A heavy chain of length l is held by its upper end, so that its lower end is at a height l above a horizontal plane. If the upper end is let go, prove that at the instant when half the chain is coiled up on the plane, the pressure on the plane is to the weight of the chain in the ratio 7 : 2.

32. A heavy chain of length l, and weight w, is suspended from one end, and the lower end is at the height l above a horizontal table. If the upper end be let go, prove that as the chain falls on the table the pressure on the table increases from $2w$ to $5w$. If after the chain has fallen one end of the coil be taken hold of and lifted with uniform velocity, find how the requisite lifting force changes until the coil is completely lifted off the table.

33. A chain of length l is coiled up at the edge of a table. One end is fastened to a particle whose mass is the same as that of the whole chain. The other end is put over the edge. Prove that immediately after leaving the table the particle is moving with speed $\sqrt{\frac{1}{6}gl}$.

34. A number n of particles, each of mass p, are attached at equal intervals a to one portion of a string, and are heaped up upon a horizontal table; the other portion of the string is passed over a small smooth pulley vertically above the heap at a distance greater than na from it, and carries a mass m at its free end; at the commencement of motion the first particle is just leaving the table; show that

$$3(m+np)^2v^2=ga\{6(n-1)m^2-n(n-1)(2n-1)p^2\},$$

v being the speed of m just after the last particle leaves the table.

35. Three particles A, B, C are in a straight line attached to points on a string and are moving in a plane with equal velocities at right angles to this line, their masses being m, m' and m respectively. If B comes in contact with a fixed obstacle whose restitution is perfect, prove that the initial radius of curvature of the paths which A and C begin to describe is $\frac{a}{2}$, where $AB=BC=a$.

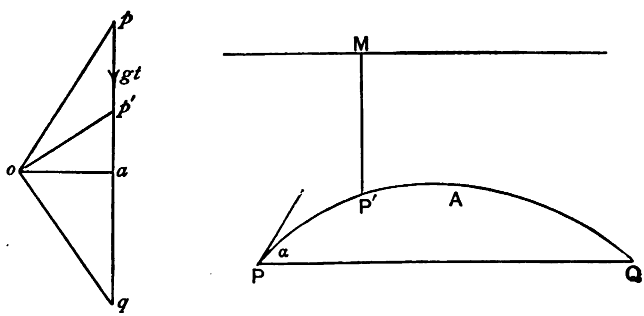
CHAPTER VII.

SPECIAL PROBLEMS. PROJECTILES.

152. A PARTICLE projected in any direction *in vacuo* is acted on by its weight (mg), and by no other force. We have already remarked (§ 64) that in consequence it has an acceleration g vertically downwards, independent of the velocity of projection, and that the path is therefore a parabola with its axis vertical. The latus rectum is $\frac{2u^2}{g}$ (§ 40, iii.), where u is the constant horizontal component of the velocity.

153. Summary of Kinematical Formulae connected with the Motion.

(1) *Velocities.* Let PAQ represent the path, and let pq be the hodograph, o its pole, and let small letters in the hodograph correspond to large letters in the path. Let op, the initial velocity, be of magnitude V, inclined at an angle α to the horizontal.



Let A be the highest point of the path. Draw oa perpendicular to pq. Then oa, numerically equal to $V \cos \alpha$, repre-

sents the constant horizontal component of the velocity which is also the velocity at A.

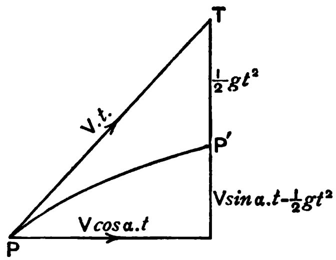
Let P' be the position of the particle t seconds after starting. Then op' representing the velocity at P', $pp' = gt$.

Thus the velocity at P' has for horizontal and vertical components $V\cos\alpha$ and $V\sin\alpha - gt$ respectively.

It is inclined to the horizon at an angle $\tan^{-1}\dfrac{V\sin\alpha - gt}{V\cos\alpha}$.

The hodograph shows that the velocities at equal intervals before and after reaching A [that is, at equal vertical heights] are equal in magnitude and equally inclined to the horizon in opposite directions.

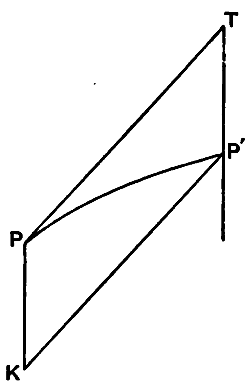
(2) *Displacements.* P' being the position of the particle at time t, the displacement PP' is equal to the vector sum of



(i.) a displacement Vt in the direction of V, and a displacement $\frac{1}{2}gt^2$ vertical; or

(ii.) a horizontal displacement $V\cos\alpha \,.\, t$ and a vertical upward displacement
$V\sin\alpha \,.\, t - \frac{1}{2}gt^2$.

(3) PT being the tangent at P, $P'T$ vertical through P', complete the parallelogram $PKP'T$.



Then $\quad PT = Vt,$

$$TP' = \tfrac{1}{2}gt^2;$$

$$\therefore \quad PT^2 = \frac{2V^2}{g} \,.\, TP',$$

or $$P'K^2 = \frac{2V^2}{g} \,.\, PK.$$

Now PK is the diameter through P, and $P'K$ is an ordinate to it, being parallel to the tangent at P. Therefore, by a well-known property of the parabola, $P'K^2 = 4SP \,.\, PK$, where SP is the focal distance of P.

Hence $\dfrac{2V^2}{g} = 4SP$, or $V^2 = 2g \,.\, SP$.

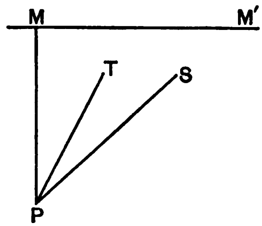
Hence the speed at P is that due to a fall from a point on the directrix vertically above P.

Since any point may be regarded as the point of projection, **the speed at any point of the path is that which would be acquired by a particle falling freely from rest at the directrix to that point.**

For instance (see the first figure) the speed at P' is equal to $\sqrt{2g \, . \, P'M}$, where $P'M$ is the perpendicular from P' on the directrix.

Example. Deduce the value of the latus rectum from this property.

(4) We can now construct the focus and directrix of the path geometrically.



From P draw PM vertical and equal to $\frac{V^2}{2g}$.

A horizontal line MM' through M is the directrix.

Through P draw PS, making an angle SPT with the direction of projection PT equal to angle TPM. Make $SP = PM$. Then S is the focus.

154. Range and Time of Flight on a Horizontal Plane through the point of projection. To obtain the time of flight we have only to equate the vertical displacement [§ 153 (2)] to zero.

Thus $V \sin \alpha \, . \, t - \frac{1}{2}gt^2 = 0$.

The solution $t=0$ corresponds to the point of projection.

Hence the time of flight $= \dfrac{2V \sin \alpha}{g}$, and the range

$$= \text{horizontal displacement in this time}$$

$$= V \cos \alpha \, . \, \frac{2V \sin \alpha}{g} = \frac{V^2 \sin 2\alpha}{g}.$$

This may also be written $\dfrac{2uv}{g}$, where u, v are the horizontal and vertical resolved parts of the initial velocity.

Corollary. For a given speed of projection the range is a maximum when $\sin 2\alpha = 1$, *i.e.* $\alpha = \dfrac{\pi}{4}$, and the range then

$$= \frac{V^2}{g} = 4(\text{greatest height to which the projectile attains}).$$

Further, if the range R is given, we have the equation $\sin 2a = \frac{Rg}{V^2}$ to determine the direction of projection. We may distinguish three cases.

(1) If $\frac{Rg}{V^2} > 1$, *no* angle of projection will enable the projectile to reach the required range.

(2) If $\frac{Rg}{V^2} = 1$, *one* angle of projection will enable the projectile to do so. The value of this angle is $\frac{\pi}{4}$.

(3) If $\frac{Rg}{V^2} < 1$, we can find two values of $2a$ less than π satisfying the equation $\sin 2a = \frac{Rg}{V^2}$.

Let these be $2a_1$, $2a_2$. These are supplementary;

$$\therefore \quad 2a_1 + 2a_2 = \pi,$$

or

$$\frac{a_1 + a_2}{2} = \frac{\pi}{4},$$

which shows that the possible directions of projection are equally inclined to the direction for greatest range.

The case in which the particle, instead of being projected freely, is projected *along the surface* of a smooth plane presents no difficulty. If i be the inclination of the plane to the horizon, we have only to write $g \sin i$ for g in the investigations of the two preceding articles.

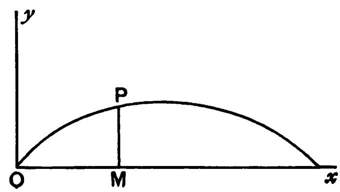
155. Cartesian Equation to the Path. Let O be the point of projection; take O for origin, and let the axis of x be horizontal, that of y vertical. Then if (x, y) be the coordinates of the particle at time t after projection,

$$x = V \cos a \,.\, t,$$

$$y = V \sin a \,.\, t - \tfrac{1}{2} g t^2;$$

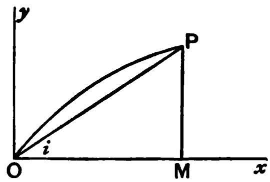
whence, eliminating t,

$$y = x \tan a - \frac{g x^2}{2 V^2 \cos^2 a}.$$



156. Range on an Inclined Plane through the Point of Projection at right angles to the vertical plane in which the path lies. Taking the same axes as before, the equation to the path is

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}.$$



Let i be the inclination of the inclined plane. The equation to the line in which it cuts the vertical plane of projection is

$$y = x\tan i.$$

Eliminating y between these equations, we have

$$\frac{gx^2}{2V^2\cos^2\alpha} = x(\tan\alpha - \tan i),$$

an equation giving the abscissae of the two points in which the path cuts the inclined plane.

The solution $x=0$ clearly gives the abscissa of O, the point of projection.

Accordingly, if OP be the range, OM the abscissa of P,

$$OM = x = \frac{2V^2\cos^2\alpha(\tan\alpha - \tan i)}{g}.$$

Hence the range $= OM\sec i = \dfrac{2V^2}{g}\dfrac{\cos\alpha\sin(\alpha - i)}{\cos^2 i}.$

Corollary. The value of the range may be written

$$\frac{V^2}{g\cos^2 i}\{\sin(2\alpha - i) - \sin i\}.$$

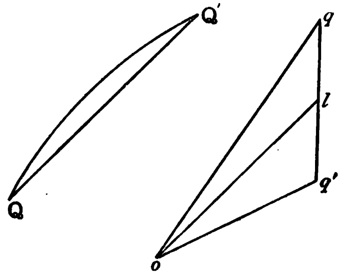
From this may be deduced, as in § 154:

(i.) That if V and i be given, the range is a maximum when $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + i\right)$, that is, when the direction of projection bisects the angle between the vertical and the line of greatest slope.

(ii.) That corresponding to any particular value of the range short of the greatest, there are two values α_1 and α_2 of the angle of projection, such that $\frac{\alpha_1 + \alpha_2}{2} = \frac{1}{2}\left(\frac{\pi}{2} + i\right)$; and consequently the corresponding directions of projection are equally inclined to the direction for maximum range.

As we shall give an independent proof of these important properties in the next article, we leave the above deductions as an exercise to the student.

157. *To find from the hodograph the range of a projectile on a given inclined plane through the point of projection at right angles to the vertical plane of the path, and to discuss the circumstances of projection.*



(1) **Range.** Let QQ' be the range on the inclined plane, qq' the hodograph for the portion of the path QQ', o the pole of the hodograph, l the middle point of qq', t the time from Q to Q'.

Then (§ 40, ii., Cor.) ol is the average velocity from Q to Q'.

Therefore the range $= QQ' = ol \, . \, t$.

But $ql = lq' = \dfrac{gt}{2}$, and therefore $t = \dfrac{2ql}{g}$.

Hence the range $= 2 \, . \, \dfrac{ol \, . \, lq}{g} = \dfrac{2uv}{g}$, *where* u, v *are the magnitudes of the (oblique) components of the initial velocity parallel to the plane and the vertical.*

This form of the expression for the range, which is convenient to remember, may be at once reduced to that of the last article.

We have, in fact, $oq = V$, $ol = u$, $lq = v$, angle $qol = \alpha - i$, angle $oql = \dfrac{\pi}{2} - \alpha$, angle $olq = \dfrac{\pi}{2} + i$.

$\therefore$ by trigonometry,

$$u = \frac{V \cos \alpha}{\cos i}, \quad v = \frac{V \sin(\alpha - i)}{\cos i},$$

and
$$\frac{2uv}{g} = \frac{2V^2}{g} \cdot \frac{\cos \alpha \sin(\alpha - i)}{\cos^2 i}.$$

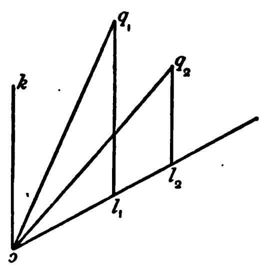
(2) Maximum range for a given speed.

Since ol is parallel to QQ', the angle $olq=\frac{\pi}{2}+i$, and is therefore constant for all angles of projection. Hence the range $\left(=\frac{2ol\,.\,lq}{g}\right)$ is proportional to the area of the triangle olq. Now oq the speed being given, this area is a maximum when $ol=lq$, and the triangle olq is isosceles. Hence the range is a maximum when the direction oq of projection is equally inclined to the plane and the vertical. Further, since in this case $ol=lq=q'l$, angle qoq' is a right angle. Hence the tangents to the path at Q, Q' are at right angles, and QQ' is a focal chord of the path. Also, the maximum range

$$=ol\,.\,t=\tfrac{1}{2}qq'\,.\,t=\tfrac{1}{2}gt^2.$$

Corollary. It is evident that if s be the length of any focal chord of the path, t the time in the portion of the path cut off by it, $s=\frac{1}{2}gt^2$.

(3) Angles of projection for a given range.



Let oq_1 be an initial velocity which gives a range corresponding to the area ol_1q_1; take on ol_1, produced if necessary, a length ol_2 equal to l_1q_1, and draw from l_2 an upward vertical l_2q_2 equal to ol_1. The triangles ol_1q_1, q_2l_2o are equal in all respects. Thus the initial velocities oq_1, oq_2, which are numerically equal, will give the same range.

Let ok be the vertical through o. Then angle koq_1=angle oq_1l_1=angle q_2ol_2. Therefore the directions of projection oq_1, oq_2 are equally inclined to the line of greatest slope and the vertical, or what comes to the same thing, to the direction of greatest range.

Examples.

[The student will find that the use of the hodograph gives an instantaneous solution of most of these examples.]

1. If Q, Q' be two points on the path of a projectile, and P the point of the path at which the tangent is parallel to QQ', then the time from Q to P is half that from Q to Q'.

2. If the tangents at Q, Q' intersect at T, the speeds at Q, Q' are proportional to QT, TQ'.

3. Given the range R and the time of flight t, find the angle of projection and the speed of projection.

$$\left[\text{Angle of projection} = \tan^{-1}\frac{gt^2}{2R}.\right.$$

$$\left.\text{Speed of projection} = \left\{\left(\frac{R}{t}\right)^2 + \left(\frac{gt}{2}\right)^2\right\}^{\frac{1}{2}}.\right]$$

4. The tangent of the angle of inclination to the horizon of the velocity at any point of the path is proportional to the time the particle takes to travel from the given point to the highest point.

5. A particle is projected at an angle α to the horizon with speed g. Prove that its direction of motion is inclined to the horizon at an angle $\frac{\alpha}{2}$ after $\tan\frac{\alpha}{2}$ seconds, and at $\frac{\pi-\alpha}{2}$ after $\cot\frac{\alpha}{2}$ seconds.

6. A particle is projected with given speed. Find the elevation of projection (α) so that its direction of motion may be horizontal when it strikes a given inclined plane (i) through the point of projection.

[With the notation of § 157, we are to have angle $oq'l$ a right angle; hence since $q'l = lq$, $\tan\alpha = 2\tan i$, or $\alpha = \tan^{-1}(2\tan i)$.]

Obtain graphic solutions to the two following:

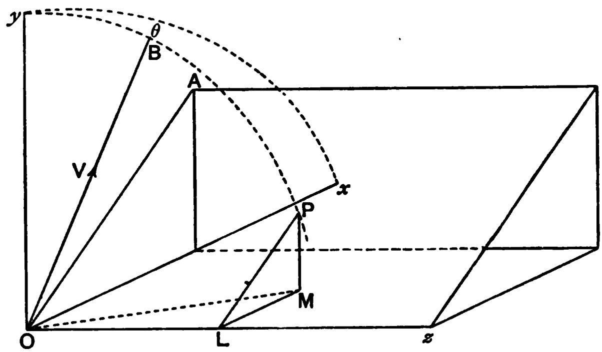
7. If t_1, t_2 be the two times of flight corresponding to any given range on a plane of inclination i, then $t_1^2 + t_2^2 + 2t_1t_2\sin i$ is independent of i, the initial speed being given.

8. If two particles are projected from a given point with a given speed V, their directions of projection being in the same vertical plane, and if t, t' be their times of flight on the horizontal plane through their common point of projection and T, T' their times to the other common point of their paths, show that $tT + t'T'$ is independent of the direction of projection.

158. Though graphic methods are often instructive, it will more frequently be found advisable to **consider the motion resolved along some chosen line.** To illustrate the method we will solve the following example:

A particle is projected in any manner from a point O *of an inclined plane, and strikes the plane again above the point of projection; find the range on the plane, and if the particle be without restitution, find the time which elapses before it returns to the level of the point of projection.*

Let OA be the line of greatest slope through O. Take Ox its horizontal projection for axis of x, and let the inclination of the plane, which is equal to the angle AOx, be i.



Let the axis of y be vertical, and the axis of z a horizontal line on the plane.

Let P be the point at which the particle strikes the plane. Draw PM perpendicular to the plane zOx, ML perpendicular to Oz. Then PL is also perpendicular to Oz. (Euclid XI. 11.)

$$\therefore \text{ angle } PLM = i.$$

Let θ be the angle between the plane of the path and the plane yOx.

$$\therefore \text{ angle } OML = \text{angle } MOx = \theta.$$

We thus have $$\frac{PM}{LM} = \tan i, \quad \frac{LM}{OL} = \cot\theta,$$

and therefore $$\frac{PM}{OL} = \tan i \cot\theta.$$

$$\therefore\quad OP^2=PM^2+ML^2+OL^2=OL^2(\tan^2 i\cot^2\theta+\cot^2\theta+1),$$

or
$$OP=OL\frac{\sqrt{1+\tan^2 i+\tan^2\theta}}{\tan\theta}.\quad\ldots\ldots\ldots\ldots(1)$$

Now let V be the velocity of projection, in direction OB, and let the angle BOM which OB makes with the horizontal plane be α.

The resolved parts of V are

$$V\cos\alpha \text{ parallel to } OM,\quad V\sin\alpha \text{ parallel to } Oy;$$

that is,

$$V\cos\alpha\cos\theta \text{ parallel to } Ox,\quad V\sin\alpha \text{ parallel to } Oy,$$
$$V\cos\alpha\sin\theta \text{ parallel to } Oz.$$

Consequently the resolved velocity normal to the inclined plane is
$$V\sin\alpha\,.\cos i-V\cos\alpha\cos\theta\,.\sin i.$$

The resolved acceleration in this direction is $-g\cos i$. Consequently, if t be the time which elapses before the particle strikes the plane, we have, equating the displacement normal to the plane to zero,

$$V(\sin\alpha\cos i-\cos\alpha\cos\theta\sin i)t-\tfrac{1}{2}g\cos i\,.\,t^2=0,$$

whence
$$t=\frac{2V}{g}\,\frac{\sin\alpha\cos i-\cos\alpha\cos\theta\sin i}{\cos i}.$$

The velocity resolved parallel to Oz remains constant. Hence

$$OL=V\cos\alpha\sin\theta\,.\,t=\frac{2V^2}{g}\,.\,\frac{\cos\alpha\sin\theta(\sin\alpha\cos i-\cos\alpha\cos\theta\sin i)}{\cos i}.$$

And, making use of (1), the range OP

$$=\frac{2V^2}{g}\,.\,\frac{\cos\alpha\cos\theta(\sin\alpha\cos i-\cos\alpha\sin i\cos\theta)}{\cos i}\sqrt{1+\tan^2 i+\tan^2\theta},$$

which reduces when $\theta=0$ to the expression obtained in the preceding article.

Next, the resolved velocity parallel to OA is unaltered by the impact of the particle on the plane. Its initial value is

$$V\sin\alpha\sin i+V\cos\alpha\cos\theta\cos i,$$

and the resolved acceleration in this direction is $-g\sin i$. Consequently, if t now represent the time from the beginning of the motion till the particle again reaches Oz, we have

$$V(\sin\alpha\sin i+\cos\alpha\cos i\cos\theta)t-\tfrac{1}{2}g\sin i\,.\,t^2=0,$$

whence
$$t=\frac{2V}{g}\,.\,\frac{\sin\alpha\sin i+\cos\alpha\cos i\cos\theta}{\sin i}.$$

Example. Find the ratio of the latera recta of the two parabolas described in the above example.

159. *A number of particles are projected from the same point with the same speed, but in different directions, all in a vertical plane. To find the* **envelope of the paths.**

Let a be the angle of projection of one of the particles. The equation of the path of this particle is

$$y = x \tan a - \frac{gx^2}{2V^2 \cos^2 a},$$

or

$$y = x \tan a - \frac{gx^2}{2V^2}(1 + \tan^2 a).$$

Regarding this as a quadratic in $\tan a$, and expressing the condition that the quadratic has equal roots, we obtain for the equation to the envelope

$$x^2 = 4 \cdot \frac{gx^2}{2V^2}\left(y + \frac{gx^2}{2V^2}\right),$$

or

$$\frac{V^2}{2g} = y + \frac{g}{2V^2}x^2,$$

rejecting the irrelevant factor x^2.

Now one of the particles may be supposed projected vertically upwards; if h be the height to which this particle rises, $V^2 = 2gh$, whence the equation to the envelope becomes

$$h = y + \frac{x^2}{4h},$$

or

$$x^2 + 4h(y - h) = 0,$$

a parabola, with latus rectum $4h$, axis vertical, and focus at the point of projection.

If the directions of projection are not confined to one plane, the envelope of the paths is a paraboloid of revolution of which the parabola just determined is the generating curve.

This result explains the form taken by a fountain-jet which consists of a number of separate jets of water projected from neighbouring points with sensibly equal speed. The more oblique jets are usually absent.

160. The student will find the following examples on the enveloping parabola suggestive. The possible directions of projection are supposed to be confined to one vertical plane, and the 'inclined plane' which the particle strikes is thus represented by a *line* in this vertical plane.

(1) Prove that the maximum range for a particle projected with given speed from a given point on an inclined plane through the point of projection is the intercept made by the enveloping parabola on the plane.

This is evident, since no point *outside* the enveloping parabola can be reached with the given speed.

(2) Prove that if S be the focus of the enveloping parabola, S' that of one of the paths, P its point of contact with the envelope, S, S' and P are collinear.

This at once appears from the fact that, for a maximum range on a given inclined plane, the inclined plane is a focal chord of the path. (See § 157.)

The student may also obtain an independent analytical solution.

(3) The particle being projected with given speed from one given point, to find the maximum range on an inclined plane of given inclination through another given point.

Let the coordinates of the second given point referred to horizontal and vertical axes through the point of projection be a, b. The equation of the plane may be written

$$\frac{x-a}{\cos\theta}=\frac{y-b}{\sin\theta}=r,$$

θ being its inclination to the horizon.

Where this plane cuts the enveloping parabola

$$x^2+4hy=4h^2,$$

we have $\qquad x=a+r\cos\theta,\quad y=b+r\sin\theta,$

and therefore $(r\cos\theta+a)^2+4h(r\sin\theta+b)-4h^2=0.$

The roots of this quadratic in r give the maximum ranges up and down the inclined plane.

161. Relative Motion of Projectiles. If two particles A and B have at a given instant the respective velocities $\bar{u}$ and $\bar{u}'$, and if they have the respective accelerations $\bar{f}$ and $\bar{f}'$, then their displacements in an interval t reckoned from the given instant are respectively

$$\bar{s}=\bar{u}t+\tfrac{1}{2}\bar{f}.t^2,\quad \bar{s}'=\bar{u}'.t+\tfrac{1}{2}\bar{f}'.t^2.$$

Consequently the displacement of B relative to A is given by

$$\sigma=\bar{s}'-\bar{s}=(\bar{u}'-\bar{u})t+\tfrac{1}{2}(\bar{f}'-\bar{f})t^2,$$

or the motion of B relative to A is a parabolic motion, with a velocity equal to the initial velocity of B relative to A, and an acceleration equal to the acceleration of B relative to A.

In the particular case of motion under gravity we have $\bar{f}=\bar{f}'=\bar{g}$; and the relative path becomes $\bar{\sigma}=(\bar{u}'-\bar{u})t$, a straight line in the direction of the relative velocity, which is in this case constant.

162. When the resistance of the air is taken into account, the path of a projectile departs widely from a parabola; the maximum range for cannon shot is in practice obtained at an elevation of about 33°; the chief characteristic of the path is in its latter portion, which is very much steeper than it would be *in vacuo*, thus reducing very largely the 'dangerous space' or space in which a man of normal height would be liable to be hit. The mathematical treatment of resisted motion is beyond the limits of this treatise.

Examples on Chapter VII.

1. A cannon-ball has a range R on a horizontal plane. If h and h' are the greatest heights in the two paths for which this is possible, show that $R=4\sqrt{hh'}$.

2. Determine the directions in which a projectile may be thrown from a given point with given speed in order that it may pass through another given point.

Show that if the second point is distant x horizontally and y vertically upwards from the first, the least possible value of the given speed is $\sqrt{gy+g\sqrt{x^2+y^2}}$.

3. Two smooth planes are at right angles with their edge of intersection horizontal, and are equally inclined to the horizon. Show that a particle, whose restitution is perfect, if projected horizontally in a direction perpendicular to the common edge from a point vertically above it will return to its original position after two rebounds.

4. A body is projected along the surface of a smooth sphere of radius r from the highest point with velocity $\frac{1}{2}\sqrt{gr}$; find where it will leave the surface, and prove that it will strike a horizontal plane through the centre of the sphere at a distance from the centre

$$r\frac{9\sqrt{39}+7\sqrt{7}}{64}.$$

5. From a point between two parallel walls, and at distances b, c from them, a smooth ball, coefficient of restitution e, is thrown, and it returns to the point of projection after one rebound at each wall, show that its range on the horizontal plane through the point

of projection would, if undisturbed, have been equal to one or other of the expressions

$$\left(b+\frac{c}{e}\right)\left(1+\frac{1}{e}\right), \quad \left(c+\frac{b}{e}\right)\left(1+\frac{1}{e}\right).$$

6. A particle tied to a light inextensible string of length l is projected horizontally with a speed V from the point to which it is attached. Show that the least energy will be lost by the impulse when $V^2=lg/\sqrt{3}$.

7. A man travelling round a circle of radius a at speed v throws a ball from his hand at height h above the ground with a relative velocity V, so that it alights at the centre of the circle. Show that the least possible value of V is given by

$$V^2=v^2+g(\sqrt{a^2+h^2}-h).$$

8. Find the least speed u with which a particle can be projected from a point on the ground so that it may pass over a given wall, and find the speed u' with which the particle will in this case reach the wall.

Prove that if the particle is projected with speed v ($>u$) the length of the top of the wall which can be reached by the particle is

$$2\frac{\{(v^2-u^2)(v^2+u'^2)\}^{\frac{1}{2}}}{g}.$$

9. Two smooth planes of height h whose inclinations to the horizon are α and β are placed back to back. A particle is projected from the foot of one of them in a direction making an angle γ with the line of greatest slope so as to move along its surface and just to pass over the common section of the two planes. Find the distance the particle will travel parallel to the common section before reaching the foot of the second plane.

10. A body is projected with speed u from a fixed point at a distance a from a fixed plane inclined at an angle α to the horizon so as to strike the plane at right angles. Prove that if the vertical plane in which the projection takes place cuts the inclined plane along the line of the greatest slope,

$$u^2>ag(\sqrt{4-3\cos^2\alpha}-\cos\alpha).$$

11. If a parabola be placed in a vertical plane with its axis vertical and vertex downwards; and if a smooth particle, whose restitution is perfect, be projected from the focus in the plane of the parabola with any speed in any direction so as to strike the parabola and to rebound; prove that, in each of its successive rebounds, it will pass through the focus.

12. A bullet is projected with velocity V at any acute angle α greater than the least positive value of $\sec^{-1}3$; show that its path

will cut two planes through the point of projection perpendicularly; that if their inclinations to the horizon are β and γ, then $\beta+\gamma=\alpha$; and that the time it takes in passing from the one plane to the other is equal to

$$\frac{V}{g}\sin(\beta-\gamma).$$

13. A particle of restitution e is projected directly up an inclined plane of elevation α in a direction making an angle β with the plane. Prove that the particle will return to the point of projection during its parabolic motion if

$$\frac{\log\{1-(1-e)\cot\alpha\cot\beta\}}{\log e}$$

be an integer.

14. A shot is fired with speed $\sqrt{2gh}$ from the top of a mountain which is in the form of a hemisphere of radius r. Show that the furthest points of the mountain which can be reached by the shot are at a distance (measured in a straight line) $r-\sqrt{r^2-4rh}$ from the point of projection.

15. A fort of vertical height h stands on a plane hill-side which makes an angle α with the horizon. Show that a gun that can fire with muzzle velocity V from the top of the fort commands a district whose shape is an ellipse of eccentricity $\sin\alpha$, and whose area is

$$\pi\sec\alpha\left(\frac{V^4}{g^2}\sec^2\alpha+\frac{2V^2h}{g}\right).$$

16. A particle is projected from the highest point of a sphere of radius c, so as to clear the sphere. Prove that speed of projection cannot be less than $\sqrt{(\frac{1}{2}gc)}$.

17. Two bullets of masses m, m' which are describing parabolas of latera recta l, l' in the same vertical plane, collide and coalesce. Prove that the latus rectum of their path after impact will be

$$\left(\frac{m\sqrt{l}\pm m'\sqrt{l'}}{m+m'}\right)^2.$$

18. Two particles of masses m, m' respectively are projected with equal speeds and at the same angle in opposite directions from two points in the same horizontal plane, so that the particles strike one another. If the two coalesce, find the greatest height which the particle so formed attains, and show that the latus rectum of its path is $\left(\frac{m-m'}{m+m'}\right)^2\times$ latus rectum of the path of either of the given particles before impact.

19. Three particles are projected simultaneously in the same vertical plane from a given point with velocities v_1, v_2, v_3 in directions

making α_1, α_2, α_3 with the vertical. Show that at any instant a straight line will pass through them if

$$\Sigma\left\{\frac{\sin(\alpha_3-\alpha_2)}{v_1}\right\}=0.$$

20. A number of particles of equal restitution are projected from a point on a smooth inclined plane with equal speeds in directions making the same angle with the plane. Show that they will all cease to rebound at the same instant, and that they then all lie on a circle.

21. Two projectiles move in the same vertical plane: their distances apart when on the same horizontal line and when on the same vertical line are respectively h_1, h_2. Prove that their least distance apart is $h_1h_2/\sqrt{h_1^2+h_2^2}$.

22. Two particles are describing the same parabola under gravity. Show that the intersection of their directions of motion moves as a heavy particle in an equal co-axial parabola, the distance between the vertices of the two parabolas being $\frac{1}{8}g\tau^2$, where τ is the interval between the instants at which the two particles pass the vertex.

23. Referred to the centre of the wheel as origin and the downward vertical as axis of y, show that the envelope of the splashes of mud from the wheel on the side of an omnibus travelling with velocity V is

$$\frac{x^2}{a^2}-1=\frac{2V^2}{ag}\left(\frac{y}{a}+\frac{V^2}{2ag}\right),$$

where a is the radius of the wheel.

24. A charge of pellets projected from a gun has the form of a right cone of semi-vertical angle α with its vertex at the muzzle. If it strikes a target whose plane is vertical and perpendicular to the vertical plane through the axis of the cone, show that the area of impact on the target is less than it would have been if gravity did not act in the ratio of

$$1-\frac{hg}{2v^2}\frac{\sin 2\beta}{\cos(\beta+\alpha)\cos(\beta-\alpha)}:1,$$

h being the horizontal distance of the target from the muzzle, v the speed of projection, and β the angle of inclination to the horizon of the axis of cone.

25. At a point on the ground from which a gun is fired the elevation of the top of a tower is x°; the gun is fired at an elevation y°, and the shot strikes the tower at a point whose elevation is $1'$ less than x°: show that in order that the shot may strike the top the elevation must be increased by

$$\frac{\sec^2 x^\circ}{1+2\tan y^\circ\tan x^\circ-\tan^2 y^\circ}\text{ minutes.}$$

26. A particle is projected with a speed v in a direction making an angle α with the horizon, find the speed and direction of motion at the end of a time t.

If at the end of the time t the particle receives a small blow in a direction at right angles to the plane of its motion, a small velocity u being in consequence imparted to the particle, show that it will reach the horizontal plane through the point of projection at a distance from that point greater than if it had received no blow by

$$\frac{u^2}{4g}\{\sqrt{\tan\alpha}+\sqrt{\cot\alpha}\,.\,\tan\beta\}^2,$$

where β is the angle the direction of motion makes with the horizon at the instant of receiving the blow and higher powers of u than the square are neglected.

27. A ball is projected in any manner under the action of gravity so as to impinge on an inclined plane. Prove that the successive points of impact will all lie on a parabola; and that if lines be drawn from a point to represent in magnitude and direction the velocities of the ball just after its successive impacts the locus of their ends will be a straight line.

28. A gun-carriage is propelled with constant velocity along rails which make a given angle with the line of greatest slope of an inclined plane. If the gun be fired with constant charge in directions making the same angle with the plane, show that at any instant the area commanded on the plane is a circle.

29. Two particles of masses m, m' lying close together are connected by an inextensible inelastic string of length l. One of them is projected with a velocity whose horizontal component is u, and the string makes an angle α with the horizontal when it becomes straight. Show that $u^2 > \frac{1}{2}gl\sin\alpha\cos^2\alpha$, and that the tension of the string after the other particle leaves the ground is constant and equal to

$$\frac{1}{4}\,\frac{mm'}{m+m'}\,\frac{g^2l}{u^2}\cos^4\alpha.$$

30. A heavy particle of mass m is tied by an inextensible light string of length $\dfrac{v^2}{g\sqrt{3}}$ to another particle of mass m', and both are initially at the same point on the ground, the string being slack. The first particle is projected with speed v in a direction inclined at an angle $\cos^{-1}\dfrac{1}{\sqrt{3}}$ to the horizon. Find the latus rectum of the path of the C.G. of the particles when the string becomes tight, and show that m' starts with a speed $\dfrac{m}{m+m'}\cdot\dfrac{v\sqrt{2}}{3}$.

31. Prove that if the effect of a horizontal wind on a projectile be an acceleration f in the direction of the wind, and the effects of the resistance of the air be neglected, the latus rectum of the path of a particle projected with speed v at an angle α to the horizon in the vertical plane through the direction of the wind is

$$\frac{2v^2(g\cos\alpha + f\sin\alpha)^2}{\{f^2+g^2\}^{\frac{3}{2}}}.$$

32. A shot of mass m is discharged from a gun which together with the gun-carriage is of mass M. The gun-carriage can slide on a smooth horizontal plane. Prove that for a given charge of powder the range is a maximum for an elevation $\frac{1}{2}\cos^{-1}\frac{m}{2M+m}$ of the gun.

33. A cannon on a horizontal smooth platform when fixed and pointed in a direction inclined at an angle α to the horizontal gives a muzzle velocity of magnitude v to the shot.

If when the cannon is fired off it is free to move on the platform, find what will be the speed with which the shot leaves the cannon, supposing that the charge of powder to be the same as before, and that the mass of the powder is neglected. In this case prove that the horizontal range of the shot is

$$\frac{v^2}{g}\cdot\frac{M\sin 2\alpha}{M+m\sin^2\alpha},$$

m being the mass of the shot, and M the mass of the cannon and carriage.

34. A wedge in the shape of an inclined plane of angle α is drawn along the ground with uniform velocity V in a direction towards the angle of the wedge; a smooth ball is let fall so as to strike the plane with speed u, prove that after a time

$$\frac{2e}{g}(u+V\tan\alpha)$$

it will again strike the plane, and that the latus rectum of its path after the first impact is

$$\frac{2}{g}(1+e)^2\sin^2\alpha(u\cos\alpha+V\sin\alpha)^2,$$

when e is the coefficient of restitution between the ball and the plane.

35. A spherical particle of mass m is falling vertically and impinges with speed u on an inclined plane mass M, which is free to move along a smooth horizontal table. Show that if the particle

strikes the plane a second time, the distance of the point of impact from the edge of the plane must be greater than

$$\frac{2eu^2\sin\alpha}{g}\,\frac{(M+m)(1+e)}{M+m\sin^2\alpha},$$

where e is the coefficient of restitution, and α is the inclination of the plane.

36. A ball moving uniformly in a straight line along a horizontal plane with speed u meets a small fixed inclined plane whose intersection with the horizontal plane is perpendicular to the direction of the ball's motion; show that after a time $\frac{2u}{g}\cdot\frac{1+e}{1-e}\sin\alpha\cos\alpha$ the ball will be again moving uniformly along the horizontal plane with speed $u(\cos^2\alpha - e\sin^2\alpha)$, where e is the coefficient of restitution of both planes and α the inclination of the small plane.

37. A solid column is resting with its base on the horizontal ground. A projectile is fired at it with speed V from a point in the ground distant a from the base. Prove that the projectile will be most effective in upsetting the column if fired at an elevation α given by

$$\tan^3\alpha + \tan\alpha = \frac{2V^2}{ga},$$

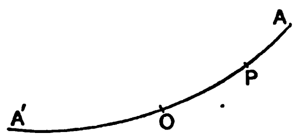
the ratio of the breadth of the column to a being neglected.

38. A particle of unit mass is moving on the inside of a smooth vertical circle under gravity and a force to the centre of the circle equal to gr/k at distance r, where k is a constant and g is the acceleration due to gravity. The particle starts from the lowest point, and after leaving the circle at some point P strikes it again at some point Q. Show that the free path is an ellipse whose centre is at a distance k below that of the circle, and that the chord PQ and the tangent at P are equally inclined to its axes.

CHAPTER VIII.

SPECIAL PROBLEMS. CYCLOIDAL AND PENDULUM MOTIONS.

163. Harmonic Motion in a Curve. Suppose that a point P is describing a portion AOA' of a curve in such a manner that the resolved acceleration along the tangent at P is always equal



to $\mu \times (\text{arc } PO)$ in the sense P to O, O being a fixed point on the curve. Let arc $OP=s$, and let v be the speed at P. The acceleration resolved along the tangent at P is equal in magnitude to what we have called (§ 35) the *speed-acceleration*, whose kinematical expression (§ 49) is $\dot{v}$ or $\ddot{s}$, and which does not involve the curvature of the path. Hence the relations between the speed acceleration ($\ddot{s}$), the speed ($\dot{s}$), and the length of path (s) are the same, whatever the curvature of the curve at P may be; hence these relations are the same as in the particular case when the curve is a straight line; but in the latter case the motion is simple harmonic.

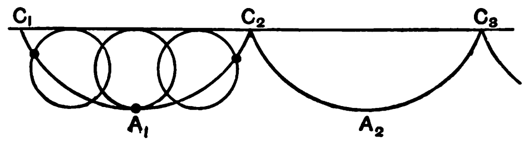
Hence, if A, A' be the extreme limits of the vibrations, and if arc $OA = \text{arc } OA' = a$,

(1) The speed of $P=\sqrt{\mu(a^2-s^2)}$.

(2) The time from A to $P=\dfrac{1}{\sqrt{\mu}}\cos^{-1}\dfrac{s}{a}$.

(3) The time of a complete vibration is $=\dfrac{2\pi}{\sqrt{\mu}}$.

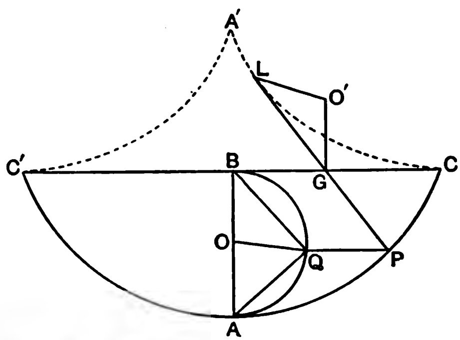
164. The Cycloid. The cycloid is a curve generated by a point fixed in the circumference of a circle, while the circle rolls without slipping on a fixed straight line.



The general shape of the curve is shown in the above figure, the position of the tracing point for the various positions of the rolling circle being indicated by a black dot. C_1, C_2, C_3, ... are called the *cusps*, A_1, A_2, ... the points of the curve at which the tangent is parallel to C_1C_3, the *vertices* of the curve.

The properties of the cycloid will be found given in any treatise on the Differential Calculus (Edwards' *Differential Calculus*, §§ 390-401). We give in the next article kinematical proofs of the properties we require, because of their compactness and intrinsic interest.

165. Let P be a point which is describing the cycloid CAC' in the sense C to A. From A, the vertex, draw AB perpendicular



to CC', and on AB as diameter describe a circle. This circle corresponds to the instantaneous position of the rolling circle when the tracing point is at A. Let O be the centre of this circle. Let a be the radius of the rolling circle, G its point of contact with CC' when the tracing point is at P, ω the circle's

angular velocity in this position. Draw PQ parallel to CC' to meet the circle AQB in Q. Join PG, BQ, QA, QO.

(1) Then, when the tracing point is at P, G is the instantaneous centre of the rolling circle.

Therefore PG is the normal to the cycloid at P, and the velocity of P in the cycloid is $\omega \,.\, PG$ (§ 52).

Hence BQ, QA are parallel respectively to the normal and tangent to the cycloid at P, and the velocity of P is equal to $\omega \,.\, BQ$ at right angles to BQ.

(2) **The arc AP of the cycloid is equal to twice the chord AQ of the circle AQB.**

For let P describe the cycloid *with constant speed* v. As PQ moves parallel to itself, OQ always remains parallel to that radius of the rolling circle which passes through the tracing point P. Hence the angular velocity of OQ = that of rolling circle $= \omega$, which latter is defined by $\omega \,.\, BQ = v$.

Therefore the velocity of Q in the circle $BQA = \omega \,.\, OQ$ perpendicular to OQ. This resolves into $\frac{1}{2}\omega \,.\, BQ$ parallel to QA, and $\frac{1}{2}\omega \,.\, AQ$ parallel to BQ. The former velocity is the rate at which AQ shortens; its magnitude $= \frac{1}{2}v$, since $\omega \,.\, BQ = v$.

Hence if t be the time which elapses while P moves from P to A,

$$\text{arc } AP = vt, \text{ and chord } AQ = \tfrac{1}{2}vt;$$

$$\therefore \text{ arc } AP = 2(\text{chord } AQ).$$

(3) **The radius of curvature of the cycloid at P is equal to twice PG.**

Let the circle roll *with constant angular velocity* ω. Its centre therefore moves with constant velocity, and is therefore (§ 53) the *acceleration centre* for all points fixed relative to the circle.

Hence (§ 53) the acceleration of $P = \omega^2 \,.\, QO$ parallel to QO.

This is equivalent to $\frac{1}{2}\omega^2 \,.\, QB$ parallel to QB and $\frac{1}{2}\omega^2 \,.\, QA$ parallel to QA.

The former is the *normal acceleration*.

Hence, if v be the speed of P, R the radius of curvature of the cycloid at P,

$$\frac{v^2}{R} = \tfrac{1}{2}\omega^2 QB.$$

But $v = \omega \,.\, QB$, and hence $R = 2QB = 2PG$.

(4) **The evolute of a cycloid is an equal cycloid.**

Produce PG to L, making $PL = 2 \,.\, PG$. L is the centre of curvature at P.

Draw LO' parallel to OQ, GO' parallel to OB, and let them meet in O'.

Then evidently $LO' = O'G = OB = a$.

As the rolling circle proceeds with constant angular velocity ω, the velocity of G or $O' = a\omega$, and since LO' is always parallel to OQ, the angular velocity of LO' is ω.

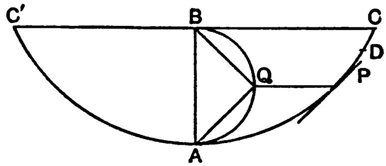
Hence L moves as if rigidly attached to a circle, also of radius a, and rolling on a line parallel to CC' and distant $2a$ from it; or the *evolute of a cycloid is an equal cycloid.*

The cusp of the latter cycloid is on AB produced.

The vertices are at C, C'.

Hence, if a string be attached at A', the cusp of this latter cycloid, and wrapped tightly round the curve, the other end being initially at C, when it is unwrapped the free end will describe the cycloid CAC'.

166. *A particle starts from rest at any point in the arc of a smooth cycloid whose axis is vertical and vertex downwards; to discuss the motion.*



Let CAC' be the cycloid, C, C' the cusps, A the vertex, AB perpendicular to CC' and equal to $2a$, AQB a circle on AB as diameter, P a position of the particle, Q the horizontal projection of P on the circumference of the circle BQA. The tangent at P is parallel to QA.

Hence the resolved acceleration of P along the tangent

$$= g\cos QAB = g\,.\,\frac{QA}{2a} = \frac{g}{4a}\,.\,s, \qquad \S\,165\ (2)$$

where s is the length of the arc AP.

The motion of P is therefore simple harmonic.

Consequently, whatever be the position on AC from which the particle starts, the time of a complete vibration is always the same, and equal to $2\pi\sqrt{\frac{4a}{g}}$; for this reason the oscillations of a particle under gravity in a cycloid are called *isochronous.*

Further, if D be the point from which the particle starts, the time from D to $P = \sqrt{\frac{4a}{g}}\,.\,\cos^{-1}\frac{arc\ AP}{arc\ AD}$.

167. Simple Pendulum. A particle attached to one end of a light inextensible string, the other end of which is fixed, the whole swinging in a vertical plane, is called a *simple pendulum.*

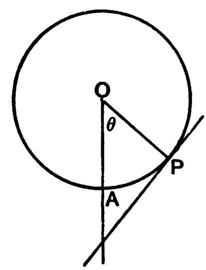
There is no such thing as a simple pendulum in nature; but it is possible to construct a pendulum which closely approximates in its behaviour to a simple pendulum by suspending a small body of great mass by means of a long fine wire. A simple pendulum which would oscillate in the same time as a given real pendulum is called the *equivalent simple pendulum* corresponding to the real pendulum.

If a simple pendulum, the suspending string of which is perfectly flexible, has its upper end attached to the cusp of the upper cycloid, shown in the figure of § 165, the string, if of proper length, will when wrapped round the curve and released describe the lower cycloid; and whatever be the amplitude of vibration, the time of a complete vibration will still be $2\pi\sqrt{\frac{4a}{g}}$.

This device is sometimes made use of for clock pendulums, the upper cycloid consisting of two steel cheeks; as the amplitude of vibration is not very great, a small portion only of the upper cycloid, symmetrical about the cusp, is required, and only the upper part of the pendulum need be flexible.

It is more usual, however, to employ a pendulum which swings over a small arc of a circle. That such a pendulum is approximately isochronous, the following investigation will show.

168. *A simple pendulum of length* l *has one end attached to a fixed point, so that the bob describes a portion of the arc of a vertical circle; to prove that the oscillations are sensibly isochronous, and that the time of a complete oscillation is* $2\pi\sqrt{\frac{l}{g}}$.



Let O be the fixed point, OA vertical, OP a position of the pendulum inclined at an angle θ to the vertical, PA a portion of the arc described.

Then the acceleration of P along the tangent $= g\sin\theta$

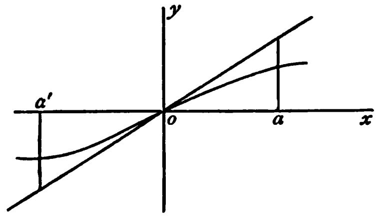
$= g\theta$, approximately, when θ is small,

$= \frac{g}{l}$ (*arc AP*).

Hence, to this approximation, the motion is simple harmonic,

and the time of a complete oscillation is independent of the amplitude and equal to $2\pi\sqrt{\frac{l}{g}}$.

169. If we take the length of arc for abscissa, the force along the tangent for ordinate, the curve of work for the



circular motion is (§ 112, Ex. 6) a portion of the sine curve, having a point of inflexion at the origin; while on the supposition that the motion is simple harmonic, it is a straight line, which is a tangent to this curve at the point of inflexion. If the pendulum be started from rest in a position inclined to the vertical, and describe an arc represented by aa', the latter supposition makes the kinetic energy at every point greater than its true value. Consequently the time of an oscillation is somewhat greater than $2\pi\sqrt{\frac{l}{g}}$, and increases with the amplitude.

In order that the pendulum, oscillating in a circle, may be sufficiently accurate for practical purposes, it is necessary that the amplitude of oscillation on one side of the vertical should not be greater than three or four degrees.

170. Observations on a pendulum afford the most accurate method of determining the value of g. So accurate indeed is the method that it is quite possible to obtain a fair result with an ordinary watch which has a second hand. A flat circular weight, such as those used for weighing letters, may be conveniently used as the bob of the pendulum; an india-rubber band should be slipped over the weight so as to lie along a diameter, and to the centre of the upper part of this may be attached a very fine thread; with a little adjustment the weight will hang with its flat sides horizontal, thus offering when in motion but little resistance to the air; the thread should be as long as possible, and may be attached to a long stout nail on the wall. The procedure is as follows: Take a white piece of paper

on which an ink line has been ruled, and place it so that from the position the observer is to occupy the thread is exactly over the ink line when the weight is hanging at rest. Then set the weight gently in motion, taking care that it swings as nearly as possible in a vertical plane.

If the length of the pendulum is, say, 7 feet, the length of the swing of the weight should not be more than about 8 inches. Now, holding the watch in the hand it will be found that by waiting a little the thread will cross the ink line as nearly as possible when the second hand of the watch is at one of the well-marked divisions, 10, 15, 20, etc. When this happens begin to count the complete oscillations of the pendulum; continue counting for five or six minutes, and end when the thread again crosses the ink mark, as the second hand as nearly as possible coincides with one of the divisions 10, 15, 20, etc. (In reckoning the number of oscillations, take care not to count in *both* the first and last oscillations.) The period of an oscillation is thus known. Measure the length (l) of the pendulum, and then calculate g from the formula

$$g=\frac{4\pi^2 l}{t^2}.$$

In an actual experiment the pendulum oscillated 170 times in 495 seconds, and its length was 6·96 feet, giving $g=32{\cdot}4$ f.s.s.

The accurate determination of g, however, involves great precautions; corrections have to be made for the fact that the pendulum is not really a simple one, for the resistance of the air and for the amplitude of the oscillation.

171. Seconds' Pendulum. A simple pendulum which in a given locality will perform half a complete oscillation in one second, is called a seconds' pendulum. To obtain the length of the seconds' pendulum in a locality where the apparent value of the acceleration due to gravity is g, we have

$$\pi\sqrt{\frac{l}{g}}=1$$

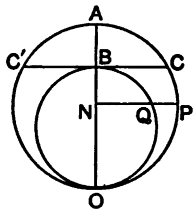
or

$$l=\frac{g}{\pi^2}.$$

The length of the seconds' pendulum in the latitude of London is 39·13929... inches, or 99·413 centimetres.

172. The motion of the bob of a pendulum may be compared with that of a bead performing complete revolutions in a vertical circle in the following way.

Let COC' be the arc of the circle described by the bob, and let OA be the vertical diameter of this circle. Join CC' cutting OA in B. On OB as diameter describe a circle. Let P denote the position of the bob at any instant. Draw PN perpendicular to OA, and let it cut the inner circle in Q. Let R and r be the radii of the outer and inner circles respectively.



Then as P *oscillates through the arc* COC', Q *moves like a particle performing a complete revolution in the inner circle starting with speed at* B *due to a height* AB, *provided that in the case of* Q *we suppose that the value of the acceleration due to gravity is reduced from* g *to* $g \cdot \frac{r^2}{R^2}$.

The proof of this we leave as an exercise to the student; he should consider the vertical velocities of the two points.*

173. If the bob of a pendulum when drawn aside is started with a small impulse which is not entirely in the vertical plane which contains the bob and the point of suspension, the pendulum will of course no longer oscillate in this vertical plane. But it is evident that as in § 168 the acceleration of the bob is approximately directed towards its equilibrium position, its magnitude being approximately $\frac{g}{l} \cdot x$, where x is its distance from that position. Hence (§ 51, Ex. 3) the motion is approximately *elliptic harmonic motion.* The path of the bob is not in reality a plane curve at all, except when it is a circle, since no ellipse can lie on a sphere. A second approximation shows that the ellipse to which the path approximates rotates in its own plane with an angular velocity proportional to its area.

* The above article is taken substantially from the *Encyclopaedia Britannica*, article *Mechanics*.

174. Let n be the number of oscillations performed by a pendulum of length l in a given time T, and let us find the change in n produced by small changes in l and g. Suppose that l becomes $l+\delta l$, and g becomes $g+\delta g$, while in consequence n becomes $n+\delta n$.

Then since $$\frac{T}{n}=2\pi\sqrt{\frac{l}{g}},$$

and $$\frac{T}{n+\delta n}=2\pi\sqrt{\frac{l+\delta l}{g+\delta g}},$$

we have $$\frac{n^2 l}{g}=\frac{(n+\delta n)^2(l+\delta l)}{g+\delta g},$$

or $$\left(1+\frac{\delta n}{n}\right)^2\left(1+\frac{\delta l}{l}\right)=1+\frac{\delta g}{g}.$$

Now suppose that $\frac{\delta n}{n}$, $\frac{\delta l}{l}$, $\frac{\delta g}{g}$ are so small that their squares may be neglected; then, multiplying out, we have

$$\frac{2\delta n}{n}+\frac{\delta l}{l}=\frac{\delta g}{g},$$

or $$\frac{2\delta n}{n}=\frac{\delta g}{g}-\frac{\delta l}{l},$$

a result which may be obtained at once from the equation

$$2n\pi\sqrt{\frac{l}{g}}=T$$

by logarithmic differentiation.

In particular, if l remains constant, $2\,.\,\frac{\delta n}{n}=\frac{\delta g}{g}$, while if g remains constant, $2\frac{\delta n}{n}=-\frac{\delta l}{l}$.

Example. *A clock keeps correct time in London, where* g $=32\cdot191$. *How much will it gain or lose in a day if taken to Paris where* g $=32\cdot183$?

We may suppose that the clock beats seconds.

Then

$$\frac{2(\text{number of seconds gained in one day})}{24\times60\times60}=\frac{32\cdot183-32\cdot191}{32\cdot191}$$
$$=-\frac{0\cdot008}{32\cdot191}.$$

The clock therefore *loses* $\frac{\cdot008\times24\times60\times60}{2\times32\cdot191}$ seconds, or about $10\cdot8$ seconds in one day.

Examples.

1. A particle slides down a smooth cycloidal arc whose plane is vertical and vertex downwards, starting from rest at the cusp. Prove that the time of falling down the first half of the vertical height is equal to the time down the second half of the vertical height.

2. A particle starts from rest at the cusp of a smooth cycloidal arc whose axis is vertical; show that when it has fallen through half the distance measured along the arc to the vertex $\frac{2}{3}$ of the time of descent will have elapsed.

3. If a particle slide down a smooth cycloid starting from a point whose arcual distance from the vertex is b, then its speed at any time t is $\frac{2\pi b}{\tau}\sin\frac{2\pi t}{\tau}$, where τ is the time of a complete oscillation of the particle.

4. A particle slides down a cycloid whose base is horizontal and vertex downwards starting from rest at the cusp. Prove from the hodograph that the acceleration of the particle is of constant magnitude.

5. A heavy particle, suspended by a string, is made to move in a cycloid by means of metal cheeks: if the extent of the oscillations is as large as possible the angular velocity of the string will be constant.

Examples on Chapter VIII.

1. The length of a pendulum beating once per second is 994·59 millimetres at Edinburgh, and 995·88 millimetres at Spitzbergen. The value of g at Edinburgh is 981·6; what is it at Spitzbergen?

2. A clock gains 5 seconds a day: calculate to 5 places of decimals by what proportion of its length the pendulum (assumed to be a simple one) ought to be lengthened.

3. A weightless straight rod ABC, of length $2a$, is movable about the end A, which is fixed, and carries two particles of the same mass, one fastened to the middle point B, and the other to the end C of the rod. If the rod is held in a horizontal position and is let go, prove that its angular velocity when vertical will be $\sqrt{(6g/5a)}$, and that $5a/3$ will be the length of the equivalent simple pendulum.

4. A pendulum which undisturbed has a period τ is placed in the vicinity of a mountain and has then a period τ', and when at rest is deflected from the vertical through an angle θ; prove that the force due to the attraction of the mountain is to the weight of the bob as

$$\{\tau^4 + \tau'^4 - 2\tau^2\tau'^2\cos\theta\}^{\frac{1}{2}} : \tau'^2.$$

5. A particle hangs vertically by a string of length a in a railway carriage moving on a straight horizontal line with uniform velocity V; prove that if the brakes are applied so as to exert a uniform retardation sufficient to bring the train to rest after running a distance b, the string will swing backwards and forwards through an angle $2\tan^{-1}(V^2/2bg)$, and that the time of oscillation will be the same as that of a simple pendulum of length $2abg(4b^2g^2+V^4)^{-\frac{1}{2}}$ swinging through the same angle.

6. Two clocks identical in all respects are placed at opposite ends of a diameter of the earth, and at one of these points the moon is vertically overhead. Assuming the earth and moon to remain at rest, and that the mass of the earth is 80 times that of the moon, show that the clocks if started simultaneously will at the end of 24 hours differ by about $\frac{3}{10}$ of a second.

(Distance of moon = 60 times the earth's radius.)

7. A particle lies on a smooth horizontal table and is attached to a string, the other end of which is drawn along the table with a constant acceleration f. The particle is originally moving in the direction of the string, but suddenly a blow is applied to it at right angles to the direction of the string. Discuss the subsequent motion.

8. Two particles fall under gravity down an inverted cycloid, starting from rest at the cusp at a given interval of time; prove that their directions of motion make a constant angle with each other and intercept a constant length on the tangent at the vertex of the cycloid.

9. Two cycloids are placed in the same vertical plane, their vertices being downward and in the same horizontal line, and their bases being horizontal. Two particles start simultaneously one on each cycloid from points at the same height above the vertices. Show that they will next be at the same height after a time

$$\frac{4\pi\sqrt{Aa}}{(\sqrt{A}+\sqrt{a})\sqrt{g}},$$

and next after that at time

$$\frac{8\pi\sqrt{Aa}}{(\sqrt{A}+\sqrt{a})\sqrt{g}} \text{ or } \frac{4\pi\sqrt{Aa}}{(\sqrt{A}-\sqrt{a})\sqrt{g}}$$

whichever is the less, A and a being the radii of the generating circles of the cycloids.

10. A cycloid is placed with its axis vertical and vertex upwards; a particle starts from rest very near the vertex and slides down the curve; show that it will leave the curve when it has fallen through a vertical distance equal to the radius of the generating circle.

11. Two smooth balls of equal mass and coefficient of restitution e move in a cycloidal tube whose axis is vertical and vertex downwards. One ball is initially at rest at the lowest point; the other is raised up an arcual distance $2a$ from the lowest point; prove that after the n^{th} impact the balls rise to arcual distances

$$\{1-(-1)^n e^n\}a \text{ and } \{1+(-1)^n e^n\}a.$$

12. A particle slides down a smooth cycloidal tube with its axis vertical and vertex downwards from a position of rest at an arcual distance s_1 from the vertex. After a time t, and before the first particle has reached the vertex, another particle commences to slide from a position of rest at an arcual distance s_2 from the vertex on the other side of it. Prove that the arcual distance from the vertex at which the particles meet is

$$\sin\frac{2\pi t}{\tau}\Big/\left\{\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{2\cos\frac{2\pi t}{\tau}}{s_1 s_2}\right\}^{\frac{1}{2}},$$

where τ is the time of a complete oscillation in the tube.

ANSWERS TO EXAMPLES.

Examples on Chapter I. PAGE 13.

2. If α, β, γ, δ be the vectors from O to the angular points, the vector to the point of intersection is $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$.
4. A circle whose centre is the centroid of the triangle ABC.
5. $\Sigma m' : \Sigma m$.

§ 25. PAGE 20.

2. $3at^2$.
3. $pa \cos pt$.

§ 33. PAGES 31, 32.

1. 54·64 cm.s., inclined at an angle of 60° to the direction of the first-mentioned velocity.
2. 5 f.s., 5·4 f.s. nearly.
4. The speed at any point is proportional to the perpendicular from the focus on the tangent.
6. Draw BE equal and parallel to DC, and CF perpendicular to AE. Draw FG parallel to CD meeting AB in G, and draw GH parallel to FC meeting CD in H. G and H are the positions required.
9. Inclination θ to original direction of steamer's motion

$$=\tan^{-1}\frac{1-\cot\beta}{1+\cot\alpha},$$

where θ, α, β are all measured in the same sense.

11. Relative velocity $=7\sqrt{2}$ f.s.

§ 34. PAGE 34.

1. 26·4 cm.s., inclined to CB at $\tan^{-1}\cdot 866 = 40° \, 54'$.
5. The six vertices of a regular hexagon.
6. A circle, the pole on the circumference.
7. A circle, the pole on the circumference.

§ 35. Page 35.

2. 38597 nearly.

§ 36. Page 37.

3. $\frac{1}{f}\{v^2+v'^2-2vv'\cos\theta\}^{\frac{1}{2}}$ seconds.

§ 43. Pages 43, 44, 45.

5. After half a second; 21 feet from the bottom.

6. (i) After $1\frac{6}{11}$ seconds more. (ii) At a height of $23\frac{73}{121}$ feet. (iii) 34 f.s., the stones reaching the ground together.

7. $7\frac{5}{6}$ seconds before the beginning of the first second.

9. After 9 seconds. The first point has travelled 63 feet.

12. Measure from O a length OY equal to $\frac{1}{2}ft^2$, in the sense opposite to f the acceleration; Y is the required point.

13. $\frac{2u^2}{f}$, where u is the constant resolved part of the relative velocity perpendicular to the direction of the relative acceleration f.

§ 47. Page 49.

1. ·000,002,666 radians per second.

5. $\frac{1}{6}\sqrt{1+4\pi^2}$ f.s., making an angle $\tan^{-1}2\pi$, or 80° 57′, with the spoke.

7. A circle described uniformly.

8. Let θ be the angle between AB and $A'B'$. The velocity of the n^{th} centre is $(2n-1)v\tan\theta$, along the line of centres. The velocity of the n^{th} pair of joints is $v\sqrt{1+4n^2\tan^2\theta}$ inclined at an angle $\cot^{-1}(2n\tan\theta)$ to the line of centres.

§ 49. Page 52.

4. $4\frac{1}{2}$ radians per second.

6. Let r be the radius from the pole to the point P, Ω the angular velocity of the tangent at P, which is equal in this case to the angular velocity of r. Then, if v be the speed of P, $v\sin a=\Omega r$.

§ 53. Page 59.

5. The point divides the rod externally in the ratio $u:v$.

§ 54. Page 60.

1. The curve of speeds is a straight line. If u, v be the speeds at the beginning and end of an interval t, the area of the curve of speeds $=\frac{1}{2}(u+v)t=ut+\frac{1}{2}ft^2$.
2. Curve of speed-accelerations a straight line; of speeds, a parabola.
4. The sine curve.

Examples on Chapter II. Pages 62–67.

2. $\dfrac{u}{\sin\phi}$
$\times\{1-\cos^2\theta\cos^2\phi+\sin\theta\cos\theta\sin\phi\cos\phi+\sqrt{3}\,.\sin\theta\sin\phi\sin(\theta+\phi)\}^{\frac{1}{2}}$, where u is the speed of A; θ, ϕ are the interior angles of OAB, and O, C are on opposite sides of AB.
7. The lines meet in a point on the circumcircle of the triangle.
9. Time $=\dfrac{(au+bv)-(av+bu)\cos a}{V}$, and the trains are at distances $\dfrac{(av-bu)(v-u\cos a)}{V^2}$ and $\dfrac{(bu-av)(u-v\cos a)}{V^2}$ from the crossing, where $V^2=u^2+v^2-2uv\cos a$.
14. The path is a parabola with B for focus.
16. The path consists of a series of parabolic arcs, with their axes parallel to the sides of the polygon, and each touching the next.
21. If CD be the semidiameter conjugate to CP, CY the perpendicular from C on the tangent at P, the normal acceleration at P may be written $\omega^2\,.\,YC$, and the speed at P, $\omega\,.\,CD$. See p. 55, Exs. 3, 4.
22. $\omega\,.\,OQ$ at right angles to OQ, where O is the fixed point and PQ a diameter.
26. Velocity of $C=2\pi\sqrt{37}$ f.s., inclined to BA at $\cot^{-1}6$. Acceleration of $C=20\pi^2\sqrt{13}$ f.s., inclined to BA at $\tan^{-1}18$.
31. Simple harmonic motion of the same period.
33. A continuous curve resembling a series of ellipses inscribed in the same square.
37. Initial velocity of the third point is $\dfrac{u_1v_2-u_2v_1}{u_1-u_2}$, and its acceleration is $\dfrac{v_2-v_1}{u_1-u_2}f$, where u_1, v_1 and u_2, v_2 are the resolved parts of the initial velocities parallel and perpendicular to HK, and f is the given acceleration.
39. An equiangular spiral. 45. 961 feet.

§ 62. Page 76.

1. Initial momentum $=657066\frac{2}{3}$ ft.-lb.-second units.
Final momentum $=1971200$ ft.-lb.-second units.
Mass acceleration $=82133\frac{1}{3}$ ft.-lb.-second units.

2. Momentum $=981\times 10^4$ C.G.S. units.
Mass acceleration $=981\times 10^3$ C.G.S. units.

§ 63. Page 77.

2. 800 dynes.

3. 264 grams' weight, nearly.

§ 68. Page 83.

1. 12000 unit blows; 750 lbs.' weight.

2. $10\frac{5}{7}$ ft. per sec.

§ 73. Page 90.

2. Speed $=g\left\{\dfrac{ma}{mg+\lambda}\right\}^{\frac{1}{2}}$.

3. Longitudinal period $=\pi\sqrt{\dfrac{2ml}{\lambda}}$ seconds;
transverse period $=\pi\left\{\dfrac{2mal}{\lambda(a-l)}\right\}^{\frac{1}{2}}$ seconds.

§ 78. Page 98.

4. No.

§ 79. Pages 98–102.

1. 11 seconds.

4. 5120 f.s.

5. The time-average will be less.

11. $g\sin BAC$ along BC.

14. The string must be divided by the hole in the inverse ratio of the masses; least value of $c=\dfrac{(m+m')}{m\omega^2}\cdot g$, where ω is the given angular velocity.

15. Tension at a distance y from the end $=my\left(1-\dfrac{x}{l}\right)g$, where m is the *mass of unit length* of the string.

18. After the jerk, both masses move with constant speed $\dfrac{P-Q}{2P}\cdot gt$.
In the second case there will be two jerks, Q moving freely in the interval between the two, and the rest of the system moving with acceleration $\dfrac{Q}{2P-Q}\cdot g$.

19. Tensions are

$$\frac{m'[M(\mu+1)+m(\mu-1)]}{M+m+m'}\cdot g \text{ and } \frac{M[2m+m'(\mu+1)]}{M+m+m'}\cdot g,$$

where M, m are the masses of the greater and less weights.

§ 84. Page 106.

2. (i) The counterpoise will move upwards with an acceleration equal to that of the boy. (ii) The counterpoise will move upwards with a speed equal to that of the boy. (iii) The counterpoise will remain at rest.

§ 85. Page 107.

2. About 282 miles.

§ 93. Page 114.

4. The impulse must pass through the centre of mass.

6. $\frac{mga}{2h}$ and mg, where m is the mass of the carriage. Through the centre of mass.

§ 97. Page 119.

1. Let A be the point of suspension; draw AB making an angle $\tan^{-1}\frac{f}{g}$ with the vertical in the opposite sense to the acceleration of the carriage. The particle must be projected in the plane which contains AB and the string, or at right angles to this plane with an appropriate speed.

2. With velocity $a\left\{\frac{g^2+f^2}{l^2-a^2}\right\}^{\frac{1}{4}}$, at right angles to the plane containing the string and the line AB defined in the previous answer.

§ 99. Page 122.

2. About 1 hour 25 minutes. Parallel to the polar axis.

3. Axes with the Earth's centre as origin; and determined in direction by straight lines drawn thence to the fixed stars. About $11\frac{1}{2}$ stone.

Examples on Chapter III. Pages 122–132.

1. (i) Newton's. (ii) Newton's. (iii) Newton's. (iv) None at present attainable. (v) The *mass* of the sun has a meaning if we refer to Newton's axes; the *weight of the sun* is an unsatisfactory phrase.

3. The motion, after thė start, is not affected.

6. Tension $=\dfrac{I^2}{2lm}$; time $=\dfrac{2\pi lm}{I}$.

7. The square of the angular velocity $=\dfrac{g}{a}\cdot\dfrac{\cos\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\alpha+\beta}{2}\cos(\alpha-\beta)}$, where a is the radius of the sphere.

10. $\dfrac{2u^2}{g}\cdot\sin\alpha$.

13. Towards the fixed end.

19. In the second case both body and wedge would slide.

20. Common speed $=\left(\dfrac{M-m}{M+m}\right)^{\frac{3}{2}}(gl\sin\alpha)^{\frac{1}{2}}-\dfrac{2I}{M+m}$, where I is the impulse which would just break the string.

26. Acceleration $=\dfrac{m_1\cos\theta-m_2\cos\left(\dfrac{\pi}{3}-\theta\right)-m_3\cos\left(\theta+\dfrac{\pi}{3}\right)}{m_1+m_2+m_3}$, where θ is the inclination to the vertical of the side on which m_1 lies.

27. If the particle is projected *up* the board, acceleration of board $=g\left(\sin\alpha-\dfrac{m}{M}\mu\cos\alpha\right)$, where M, m are the masses of the board and particle respectively; acceleration of particle relative to kinetic axes $=g(\sin\alpha+\mu\cos\alpha)$.

28. $\tan^{-1}\left\{\dfrac{M}{M+m+m'}\right\}^{\frac{1}{2}}$.

29. About 10·2 miles an hour.

45. $\dfrac{2g\sin^2\alpha}{1+2\sin^2\alpha}$.

46. $T\cdot\left\{\dfrac{M_1}{M_1+M_2}\right\}^{\frac{1}{2}}$ seconds.

§ 109. Page 142.

3. The kinetic energy after a space $4na+x$ has been described is $\dfrac{1}{2}\dfrac{Px^2}{a}$, and after a space $(4n+2)a+x$ it is $P\left(a-\dfrac{1}{2}\cdot\dfrac{x^2}{a}\right)$, where x lies between $-a$ and $+a$.

4. Greatest tension $=2mg$, extension $=\dfrac{2mg\,x_0}{\lambda}$, where mg is the weight and x_0 is the natural length of the rod.

6. The hand must not be lowered at all. The ball returns to the hand after a time $2\left\{\sqrt{\frac{2l}{g}}+\pi\sqrt{\frac{ml}{\lambda}}-\sqrt{\frac{ml}{\lambda}}\cdot\tan^{-1}\sqrt{\frac{2\lambda}{mg}}\right\}$.

§ 111. PAGE 145.

1. l must lie nearer to a than to b.

§ 112. PAGE 147.

3. $I=m\sqrt{5gl}$, where m is the mass of the particle.

5. Draw downwards through the starting point a straight line making the angle of friction with the horizontal; this will cut the wire again in the required point.

§ 113. PAGE 149.

$180{,}000\pi^2$, or 1,776,529, foot-poundals.

§ 117. PAGES 157, 158.

5. Pressure

$$=2\lambda\cot\alpha(\sin\theta-\sin\alpha)(3\sin\theta-2\sin\alpha)\sim mg(2\cos 2\alpha-3\cos 2\theta),$$

where 4α is the angle subtended by AB at the centre, and 2θ is the angle made by the radius to the bead with the upward vertical.

6. When AB subtends an angle 2θ at the centre, the speed of A or B

$$=\left\{\frac{(m+m')ga}{m+2m'\sin^2\theta}\cdot(2\cos\theta-\sqrt{2})\right\}^{\frac{1}{2}};$$

speed of C : speed of A or B :: $2\sin\theta$: 1.

§ 120. PAGES 160, 161.

2. Stable for displacements in one sense, unstable for displacements in the other.

3. Greatest speed $=2\left\{\frac{ag(A\sim B)}{2A+B}\right\}^{\frac{1}{2}}$.

4. $mg>\lambda e^2$, where m is the mass of the particle, e the eccentricity of the ellipse, λ the modulus of elasticity.

§ 124. PAGE 164.

3. 38 shillings, nearly. 4. 15,000 lbs.' weight.

5. $27\frac{3}{11}$ ft.-lbs.

Examples on Chapter IV. Pages 171–179.

8. Velocity of $m_1 = \frac{I}{m_1} \cdot \frac{b}{a+b}$, of $m_2 = \frac{I}{m_2} \cdot \frac{a}{a+b}$, both at right angles to the rod where a, b are the distances of the point of application of I from the ends.

12. $\frac{wW'}{WW' + w(W' - W)} \cdot g$.

18. $\left\{\frac{2(Q \sin\beta - P \sin\alpha) gt}{Q + P\cos^2(\alpha+\beta)}\right\}^{\frac{1}{2}}$, where l is the length of the rod.

21. After 16 impacts. 31. Whole space described $= \frac{u^2}{2\mu g}$.

33. $v = \sqrt{gl \sin\alpha}$, where α is the inclination of the plane to the horizon.

35. 200 H.P.; 200 tons before the carriages are slipped.

37. 1246 tons' weight; $31\frac{1}{2}$ foot-tons.

38. $$\text{Speed} = \left\{u^2 + (v^2 - u^2)\frac{t}{T}\right\}^{\frac{1}{2}};$$

$$\text{acceleration} = \frac{v^2 - u^2}{2T\left\{u^2 + (v^2 - u^2)\frac{t}{T}\right\}^{\frac{1}{2}}};$$

$$\text{space described} = \frac{2}{3} v \sqrt{\frac{t^3}{T}}.$$

39. $13\frac{11}{28}$ miles an hour. 40. 2200 H.P. 41. About $2\frac{1}{2}$ H.P.

42. $$F \times \frac{H_1(m_2 + m_3) - m_1(H_2 + H_3)}{H_1 + H_2 + H_3},$$

and $$F \times \frac{(H_1 + H_2)m_3 - H_3(m_1 + m_2)}{H_1 + H_2 + H_3},$$

where m_1, m_2, m_3 are the masses of the engines in pounds, and F is the retarding force per pound.

46. $24\frac{6}{11}$ H.P.

Examples on Chapter V. Pages 189, 190.

1. $\frac{1}{126}$ lb., 42,336 ft., 21 secs. $(g = 32.)$

2. $\frac{1}{4}$ lb., $\frac{4\sqrt{2}}{275}$ ft., $\frac{1}{550}$ sec. $(g = 32.)$

3. $a^2b : c^2$. $6{\cdot}492 \times 10^{34}$ nearly. $6{\cdot}492 \times 10^{34} \times \frac{c^2}{a^2b}$.

4. Unit of mass $=\dfrac{px}{yz}$ lbs., of length $=\dfrac{121pm^2z^2}{1800rxy}$ ft., of time $=\dfrac{11pmz}{240ry}$ secs. $(g=32.)$

5. Unit of mass $=\dfrac{b}{a}$ lbs., of length $=ga^2$ ft., of time $=a$ secs.

9. $\dfrac{115\sqrt{10}}{12}$ hours $=30$ hours 18 minutes, approximately.

10. 1·825 dynes.

§ 141. Pages 194, 195.

2. If $m<m'$, the end ball of each series is driven off, the remainder coming to rest. If $m>m'$, all the balls of the second series are driven off in succession from the end, each with less speed than the preceding.

4. 8·5 cm.s., 12 cm.s.; impulse $=18$ units.

§ 147. Pages 201, 202.

3. $e=\dfrac{\text{mass of } A}{\text{mass of } B}$.   5. $\cos^{-1}\left\{\dfrac{2}{u+u'}\left(\dfrac{uu'}{1-e^2}\right)^{\frac{1}{2}}\right\}$.

6. If u be the initial speed of the impinging sphere, its final speed is $u/2^6$; the speeds of the others are $u\sqrt{3}/2$, $u\sqrt{3}/2^2$, $u\sqrt{3}/2^3$, etc., the direction of the motion of each being at right angles to the tangent to it from the centre of the next.

7. A rises to a height h, while the other ball returns to the earth with a speed due to the height h.

Examples on Chapter VI. Pages 212–217.

1. Unity.

4. The straight line being $y=x\tan\alpha$, and the axis of x horizontal, the particles that have rebounded will after a time t lie on the parabola $\{y+x\tan\alpha(1+2e)+\frac{1}{2}gt^2\}^2=2gt^2x(1+e)^2 . \tan\alpha$.

9. Ten impacts.

15. Let $ABCD$ be the billiard table; draw OK perpendicular to AB and produce it to O_1, making $KO_1=e . OK$; draw O_1L perpendicular to BC produced, and produce it to O_2, making $LO_2=e . O_1L$; draw O_2M perpendicular to CD produced, and produce it to O_3, making $MO_3=e . O_2M$; draw O_3N perpendicular to AD produced, and produce it to O_4, making NO_4 equal to $e . O_3N$. Join O_4O, cutting AB, AD in P and S;

O_3S, cutting CD in R; O_2R, cutting BC in Q; then QP produced will pass through O_1, and $OPQRSO$ is the path required. (An optical method.)

26. $\frac{1}{2}m'u^2(1-e)$, where u is the original speed of m'.

28. $2pm = e(m+m')(m-m'-p)$.

32. If v be the constant velocity, the force when a length x is lifted $= \frac{w}{l} \cdot \left(\frac{v^2}{g} + x\right)$.

§ 158. Page 228.

$\sin i : \sin^2\theta$.

Examples on Chapter VII. Pages 230–236.

4. At a distance $\frac{r}{4}$ below the highest point.

8. If h be the height of the wall, and k its distance from the point of projection, $u^2 = g(\sqrt{h^2+k^2}+h)$, $u'^2 = g(\sqrt{h^2+k^2}-h)$.

9. $2h \tan\gamma(\operatorname{cosec}\alpha + \operatorname{cosec}\beta)$.

18. $\frac{V^2\sin^2\alpha}{2g}$, where V is the speed and α the angle of projection.

30. $\frac{2}{3} \cdot \frac{v^2}{g} \cdot \left(\frac{m}{m+m'}\right)^2$.

32. Assume that the kinetic energy generated is constant for all elevations.

33. Speed $= v\left\{\frac{m(2M+m)\sin^2\alpha + M^2}{(M+m)(M+m\sin^2\alpha)}\right\}^{\frac{1}{2}}$.

Examples on Chapter VIII. Pages 246–248.

1. 982·87.

2. By ·0001157 of its length.

7. Relative to the other end of the string, the particle behaves as a simple pendulum of period $2\pi\sqrt{\frac{l}{f}}$.

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